A user's guide for higher algebra

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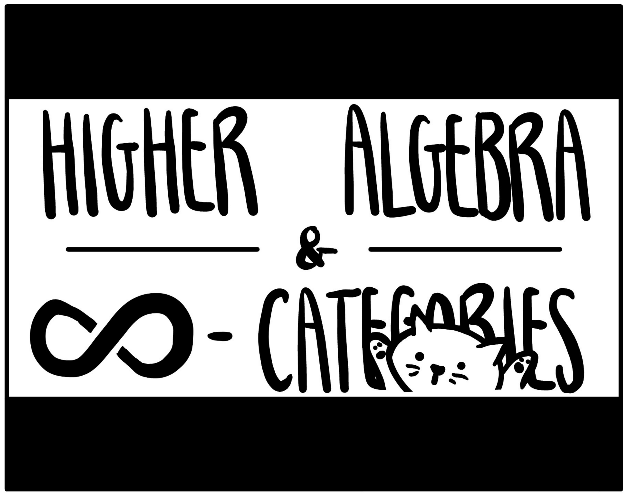
Illustrations by Maxine Calle. Part of the notes were live Tex by Thomas Brazelton. **Warning: there may be mistakes or inaccuracies!** Last edit: January 25, 2025



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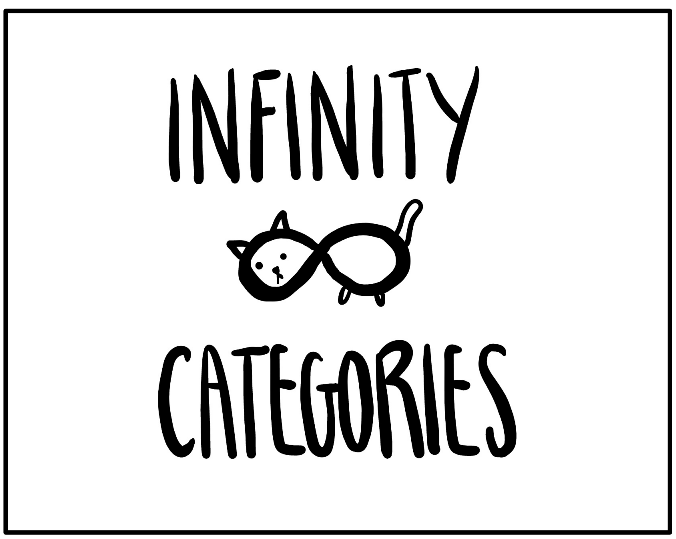
Chapter 1

Introduction

1.1 Overview

In **classical algebra**, we study sets, monoids, groups, abelian groups, rings. Each of these structures are built upon the other. In higher-level courses, we may study groupoids, which are examples of categories. Categories, more generally, can be seen as generalizations of monoids. Monoidal categories, which are categories with extra structure, are a generalization of rings, in some sense.

In **higher algebra**, we study spaces, $\mathbb{E}_1$-spaces, spectra, $\mathbf{E}_1$-ring spectra. Underlying these objects we have ∞-groupoids, ∞-categories, and monoidal ∞-categories. When we study spaces, we do not consider them up to homeomorphism, but instead up to *weak homotopy equivalence*. Thus, when we refer to "studying spaces," we will always mean that we are studying topological spaces up to weak homotopy equivalence. We now give a synthetic definition of what an ∞-category is; we will circle back to a technical definition later.



What is an ∞-category? An ∞-category (or $(\infty, 1)$-category) $\mathcal{C}$ should consist of:

1. a class of objects,
2. a class of morphisms so that $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is a space, considered up to weak homotopy equivalences
3. a class of n-morphisms for $n \geq 2$, where for instance 2-morphisms are morphisms of 1-morphisms, 3-morphisms are morphisms 2-morphisms, etc.

4. morphisms can be composed in a suitable way,

5. n-morphisms for $n \geq 2$ are invertible in some sense.

An ∞-*groupoid* (or $(\infty, 0)$-category) is an ∞-category where all the 1-morphisms are also invertible in some sense.

Why study spaces up to weak homotopy equivalence? By the Yoneda lemma, we have

$$X \cong Y \Leftrightarrow \mathrm{Hom}_{\mathrm{Top}}(A, X) \cong \mathrm{Hom}_{\mathrm{Top}}(A, Y)$$

for all $A \in \mathrm{Top}$. Figuring out $\mathrm{Hom}(A, X)$ up to bijection for all A is very difficult, so we prefer to study continuous maps up to homotopy. If X and Y are nice enough, we say that $f \simeq g$ in $\mathrm{Hom}(X, Y)$ if there is some path $I \to \mathrm{Map}(X, Y)$ so that $0 \mapsto f$ and $1 \mapsto g$. We define $[X, Y] = \mathrm{Hom}_{\mathrm{Top}}(X, Y)/\simeq$. Then $X \simeq Y$ if and only if $[A, X] \cong [A, Y]$ for all $A \in \mathrm{Top}$.

We may then ask when $[A, -] : \mathrm{Top}_* \to \mathsf{Set}$ factors through Grp or Ab. One can show that $[A, -]$ factors through Grp if and only if A is a co-H-group in Top. That is, there are maps

$$\begin{aligned} A &\to A \vee A \\ A &\to *, \end{aligned}$$

which are coassociative, counital, coinvertible.

Example 1.1.1. One example of a co-H-space is S^n for $n \geq 1$. The map $S^n \to S^n \vee S^n$ is the pinch map, and the counit is the unique map $S^n \to *$.

Definition 1.1.2. A space X is *weakly homotopy equivalent* to Y, written $X \sim Y$, if there is a map $X \to Y$ inducing an isomorphism

$$\pi_n(X) = [S^n, X]_* \cong [S^n, Y]_* = \pi_n(Y),$$

for all $n \geq 0$ (for $n \geq 1$ this is a group isomorphism).

Note that if $X \sim Y$, then $H_n(X) \cong H_n(Y)$ for any n.

Theorem 1.1.3. (Cellular approximation) For any X in Top, there exists a CW complex $\widetilde{X}$ with a canonical map $\widetilde{X} \xrightarrow{\sim} X$ that is a weak equivalence.

Theorem 1.1.4. (Whitehead) If X, Y are CW complexes, then $X \xrightarrow{\simeq} Y$ is a homotopy equivalence if and only if $X \xrightarrow{\sim} Y$ is a weak homotopy equivalence.

Exercise 1.1.5. Find spaces X and Y which are weakly homotopy equivalent but not homotopy equivalent.

1.2 Categories

1.2.1 A first definition

The definition of a category is versatile. It generalizes monoids, groups, graphs, and posets, to name a few. At the same time, the definition captures also the systematic approach in mathematics in which we introduce a mathematical structure (e.g. groups) and study a classification of all possible structures (e.g. determine all groups up to group isomorphisms).

Definition 1.2.1. A *category* $\mathcal{C}$ consists of the following data.

1. A class $\mathrm{Ob}(\mathcal{C})$. Its elements are called *objects* of the category $\mathcal{C}$. We write $X \in \mathcal{C}$ instead of $X \in \mathrm{Ob}(\mathcal{C})$.

2. For each ordered pair (X,Y) of objects in $\mathcal{C}$, we have a set $\mathrm{Hom}_{\mathcal{C}}(X,Y)$. Its elements are called *morphisms*, *arrows* or *maps from X to Y*. The set $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ is referred as the *hom-set* of X and Y in $\mathcal{C}$. An element $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$ is written as $f : X \to Y$ or as $X \xrightarrow{f} Y$ and we say $f : X \to Y$ is a map in $\mathcal{C}$. Given a map $f : X \to Y$ in $\mathcal{C}$, then X is said to be the *domain* of f and Y the *codomain* of f. We denote $\mathrm{Mor}(\mathcal{C}) = \coprod_{X,Y\in\mathcal{C}} \mathrm{Hom}_{\mathcal{C}}(X,Y)$ the class of all morphisms in $\mathcal{C}$.

3. Given objects X, Y and Z in $\mathcal{C}$, we have a function on the hom-sets:

$$\begin{array}{rcl} \mathrm{Hom}_{\mathcal{C}}(Y,Z) \times \mathrm{Hom}_{\mathcal{C}}(X,Y) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(X,Z) \\ (g,f) & \longmapsto & g \circ f \end{array}$$

   The map $g \circ f : X \to Z$ is called the *composition* or *composite of f and g*.

The above data is subject to the following axioms.

Associativity Given $f : X \to Y$, $g : Y \to Z$, and $h : Z \to W$ maps in $\mathcal{C}$, then:

$$(h \circ g) \circ f = h \circ (g \circ f).$$

Unitality For each object $X \in \mathcal{C}$, there exists a map $\mathrm{id}_X : X \to X$ in $\mathcal{C}$, called the *identity map on X*, such that:

- $\mathrm{id}_X \circ f = f$, for all maps $f : Y \to X$ in $\mathcal{C}$;
- $g \circ \mathrm{id}_X = g$, for all maps $g : X \to Y$ in $\mathcal{C}$.

The composition allows us to uniquely fill in the diagram:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, dashed, "g\circ f"'] & Y \arrow[d, "g"] \\ & Z \end{tikzcd}$$

Associativity allows to unambiguously fill the diagram below with the dotted line:

$$\begin{tikzcd} & W & \\ X \arrow[ur, dotted, "h\circ g\circ f"] \arrow[rr, dashed, "g\circ f"] \arrow[dr, "f"'] & & Z \arrow[ul, "h"'] \\ & Y \arrow[uu, dashed, "h\circ g"] \arrow[ur, "g"'] & \end{tikzcd}$$

The unitality allows us to extend any map $f : X \to Y$:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, dashed, "f"'] & Y \arrow[d, equal] \\ & Y \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, equal] \arrow[dr, dashed, "f"'] & X \arrow[d, "f"] \\ & Y \end{tikzcd}$$

Remark 1.2.2. What we have defined above is what is usually called a *locally small* category. If we require $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ to only be a class instead of set, for each pair of objects X and Y, then we say we have a *large* category.

Exercise 1.2.3. Show that the identity morphism id_X is unique in every category. In other words, given an object X, if there exists a map $\alpha : X \to X$ such that $\alpha \circ f = f$ for all maps $f : Y \to X$ in $\mathcal{C}$ and $g \circ \alpha = g$ for all maps $g : X \to Y$ in $\mathcal{C}$, then $\alpha = \mathrm{id}_X$.

Definition 1.2.4. Given a category $\mathcal{C}$, a map $f : X \to Y$ in $\mathcal{C}$ is called an *isomorphism* if there exists a map $g : Y \to X$ in $\mathcal{C}$ such that: $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$. If such g exists, it is denoted f^{-1} and is called the *inverse* of f. We say two objects X and Y in $\mathcal{C}$ are *isomorphic* if there exists an isomorphism $f : X \to Y$. In this case, we write $X \cong Y$.

Exercise 1.2.5. Show that the inverse of an isomorphism is necessarily unique.

Exercise 1.2.6. Show that composition preserves isomorphisms: given isomorphisms $f : X \to Y$ and $g : Y \to Z$ in a category $\mathcal{C}$, then $g \circ f : X \to Z$ is an isomorphism in $\mathcal{C}$. Conclude that the relation $X \cong Y$ defines an equivalence relation on $\mathrm{Ob}(\mathcal{C})$.

Example 1.2.7. The category of sets, denoted Set, is defined as follows.

1. Its class of objects are all sets. Notice here the necessity of $\mathrm{Ob}(\mathsf{Set})$ to be a class: there is no set of all sets.

2. Given sets X and Y, then $\mathrm{Hom}_{\mathsf{Set}}(X, Y)$ is the set of all functions $X \to Y$.

3. Composition in Set is defined as follows. Given $f : X \to Y$ and $g : Y \to Z$, define $g \circ f : X \to Z$ by $(g \circ f)(x) = g(f(x))$, $\forall x \in X$.

For any set X, define $\mathrm{id}_X : X \to X$ by $\mathrm{id}_X(x) = x$, $\forall x \in X$. One can check that the composition is indeed associative and unital and thus Set is a category. An isomorphism in Set is precisely a bijection.

The next examples are categories in which objects are sets with extra structure. Composition, associativity and unitality are induced by the composition in Set.

Example 1.2.8. The category of groups, denoted Grp, is defined as follows.

1. Its class of objects are groups.

2. Given groups G and H, define $\mathrm{Hom}_{\mathsf{Grp}}(G, H)$ to be the set of all group homomorphisms $G \to H$. The name hom-set originates from this category.

3. Composition is defined as in Set.

Isomorphisms in Grp are precisely group isomorphisms.

Example 1.2.9. The category of Abelian groups, denoted Ab, is defined as follows.

1. Its class of objects are Abelian groups.

2. Given Abelian groups A and B, define $\mathrm{Hom}_{\mathsf{Ab}}(A, B)$ to be the set of all group homomorphisms $A \to B$.

3. Composition is defined as in Set.

Isomorphisms in Ab are precisely group isomorphisms (between Abelian groups).

Example 1.2.10. The category of monoids, denoted Mon, is defined as follows.

1. Its class of objects are monoids.

2. Given monoids M and N, define $\mathrm{Hom}_{\mathsf{Mon}}(M, N)$ to be the set of all monoid homomorphisms $M \to N$.

3. Composition is defined as in Set.

Example 1.2.11. The category of rings, denoted Ring, is defined as follows.

1. Its class of objects are rings (with unity).

2. Given rings R and S, define $\mathrm{Hom}_{\mathsf{Ring}}(R, S)$ to be the set of all ring homomorphisms $R \to S$ (that preserves unities).

3. Composition is defined as in Set.

Isomorphisms in Ring are precisely ring isomorphisms.

Example 1.2.12. Let G be a group. Define the category of left G-sets ${}_G\mathsf{Set}$ as follows.

1. Its class of objects are left G-sets.

2. Given left G-sets X and Y, define $\mathrm{Hom}_{{}_G\mathsf{Set}}(X, Y)$ to be the set of G-equivariant maps.

3. Composition is defined as in Set.

Define similarly the category of right G-sets Set_G.

Example 1.2.13. Let $\mathbb{F}$ be a field. The category $\mathsf{Vect}_{\mathbb{F}}$ of vector spaces over $\mathbb{F}$ is defined as follows.

1. Its class of objects are vector spaces over $\mathbb{F}$.

2. Given vector spaces V and W over $\mathbb{F}$, define $\mathrm{Hom}_{\mathsf{Vect}_{\mathbb{F}}}(V, W)$ to be the set of linear transformation $V \to W$ over $\mathbb{F}$.

3. Composition is defined as in Set.

Isomorphisms in $\mathsf{Vect}_{\mathbb{F}}$ are precisely isomorphisms of vector spaces.

Warning 1.2.14. The names of the categories in the examples above can be misleading. They seem to suggest that a category is defined by its objects, but it really is not case. A category is defined by its morphisms. A better name for Set would be the "the category of set functions on all sets", and a better name for Grp would be "the category of group homomorphisms on all groups", for Ab would be "the category of group homomorphisms restricted on Abelian group", and so on. In practice, these names are too long, and it is natural to consider these mathematical objects with the appropriate morphisms.

The next examples shall emphasize how morphisms are the main actors in a category, and not objects.

Example 1.2.15. Given $(\mathbb{P}, \leq)$ a poset, it defines a category, also denoted $\mathbb{P}$, as follows.

1. $\mathrm{Ob}(\mathbb{P}) = \mathbb{P}$.

2. $\forall x, y \in \mathbb{P}$, $\mathrm{Hom}_{\mathbb{P}}(x, y) = \begin{cases} * & \text{if } x \leq y \\ \emptyset & \text{otherwise.} \end{cases}$ In other words, we write $x \to y$ if and only if $x \leq y$.

3. Composition is defined by transitivity: $\forall x, y, z \in \mathbb{P}$, if $x \leq y$ and $y \leq z$, then $x \leq z$.

We now need to check associativity and unitality.

- Associativity follows from the unambiguity of transivity: given $x \leq y$, $y \leq z$, $z \leq w$ in $\mathbb{P}$, then we can first deduce from $y \leq z \leq w$ that $y \leq w$ and thus from $x \leq y \leq w$ we obtain $x \leq w$; or we could have started from $x \leq y \leq z$ to deduce $x \leq z$, and thus from $x \leq z \leq w$ we obtain $x \leq w$.

- Unitality follows from the fact that $\forall x \in \mathbb{P}$ we have $x \leq x$. Thus we get that if $x \leq x \leq y$ then $x \leq y$. Similarly if $x \leq y \leq y$ then $x \leq y$.

Example 1.2.16. The empty set $\varnothing$ can be regarded as a poset. This defines a category $\mathbb{0}$ with no objects and no morphisms.

Example 1.2.17. Any singleton $\{\star\}$ is uniquely endowed with a poset structure. This defines a category $\mathbb{1}$ with one object and one morphism:

$$\begin{tikzcd} \bullet \arrow[loop above, "\mathrm{id}"] \end{tikzcd}$$

Since every object in a category associates an identity morphism, we often omit in the picture. Thus the category $\mathbb{1}$ is depicted as:

$$\bullet$$

Example 1.2.18. Given the poset $\{1 \leq 2\}$, we obtain a category $\mathbb{2}$ with two objects and three morphisms (two of them are identity morphisms that we omit in the picture):

$$\bullet \longrightarrow \bullet$$

Example 1.2.19. Given the poset $\{1 \leq 2 \leq 3\}$, we obtain a category $\mathbb{3}$ with three objects and six morphisms depicted as (three of them are the identity morphisms and are omitted):

$$\begin{tikzcd} \bullet \arrow[r] \arrow[rr, bend left] & \bullet \arrow[r] & \bullet \end{tikzcd}$$

or:

$$\begin{tikzcd} \bullet \arrow[r] \arrow[dr] & \bullet \arrow[d] \\ & \bullet \end{tikzcd}$$

Since the third morphism depicted is the composition of the other two, we often also omit composition, and thus we depict $\mathbb{3}$ as:

$$\bullet \longrightarrow \bullet \longrightarrow \bullet$$

Exercise 1.2.20. From the poset $\{1 \leq 2 \leq 3 \leq 4\}$, depict the category $\mathbb{4}$:

$$\bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet$$

with all its compositions.

Remark 1.2.21. Recollecting from the examples above: when we picture a category, we omit the identity and the compositions.

Example 1.2.22. The natural numbers $(\mathbb{N}, \leq)$ form a poset and thus a category $\mathbb{N}$:

$$\bullet \longrightarrow \bullet \longrightarrow \cdots \longrightarrow \bullet \longrightarrow \cdots$$

Example 1.2.23. The real numbers $(\mathbb{R}, \leq)$ form a poset and thus a category $\mathbb{R}$, but it is harder to depict than $\mathbb{N}$.

Definition 1.2.24. A category $\mathcal{C}$ is said to be *small* if $\mathrm{Ob}(\mathcal{C})$ is a set.

Example 1.2.25. If $\mathbb{P}$ is a poset, then its associated category is small. Thus $\mathbb{0}$, $\mathbb{1}$, $\mathbb{2}$, $\mathbb{3}$, $\mathbb{N}$ and $\mathbb{R}$ are small categories.

Example 1.2.26. The categories Set, Grp etc are *not* small.

Exercise 1.2.27. Show that any small categories can be regarded as a directed graph. Is every directed graph a category? What fails?

1.2.2 Monoids with many objects

The composition in a category must be associative and unital. These axioms are very similar to associativity and unitality of the binary operation of a monoid. Here we show precisely that a category is in fact a generalization of a monoid.

Example 1.2.28. Let $(M, *, e)$ be a monoid. Define a category BM as follows.

1. The category BM has a unique object, labelled arbitrarily $\star$. In other words: $\mathrm{Ob}(BM) = \{\star\}$.
2. Given that there is only one object in the category, we only need to define one hom-set. Define $\mathrm{Hom}_{BM}(\star, \star) = M$.
3. We define composition in BM via the binary operation on the monoid M:

$$\begin{aligned} M \times M = \mathrm{Hom}_{BM}(\star,\star) \times \mathrm{Hom}_{BM}(\star,\star) &\longrightarrow \mathrm{Hom}_{BM}(\star,\star) = M \\ (x, y) &\longmapsto x * y \end{aligned}$$

Since the monoid M is associative, then so is the composition on BM. The identity map $\mathrm{id}_\star : \star \to \star$ is equal to $e \in M$, the neutral element of M. Since a monoid is unital, then we obtain the unitality axiom on the category BM. Isomorphisms in BM are precisely units of M.

Remark 1.2.29. It is important to not confuse the morphisms in BM with actual set functions. Indeed, in BM, we replace entirely formally an element $x \in M$ by a map $x : \star \to \star$. But it is important to keep in mind that $\star$ is not a set and $x : \star \to \star$ is not a function of sets, it is just a different notation to express $x \in M$.

Exercise 1.2.30. Show that if $\mathcal{C}$ is a category with a unique object denoted $\star$, then $(\mathrm{Hom}_{\mathcal{C}}(\star, \star), \circ, \mathrm{id}_\star)$ is a monoid. Conclude there is a correspondence between monoids and categories with one object.

Therefore a category with many objects can be regarded as a "monoid with many objects". Perhaps the next exercise can be enlightening in that regard.

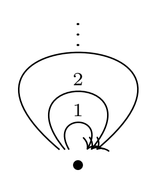
Exercise 1.2.31. Let $\{M_1, \dots, M_n\}$ be a collection of monoids. Define a category $\mathcal{C}$ as follows.

1. $\mathrm{Ob}(\mathcal{C}) = \{1, \dots, n\}$.
2. $\mathrm{Hom}_{\mathcal{C}}(i, j) = \begin{cases} M_i & \text{if } i = j \\ \varnothing & \text{otherwise.} \end{cases}$
3. Composition is induced by the binary operations on each M_i.

Verify that $\mathcal{C}$ is a category. Can you generalize this example to any collection $\{M_i\}_{i \in I}$ of monoids, where I is any set?

Example 1.2.32. Let $\{1\}$ be the trivial monoid. Then its associated category $B\{1\}$ is the small category $\mathbb{1}$.

Example 1.2.33. We can view $\mathbb{N} = (\mathbb{N}, +, 0)$ as a monoid. This defines a category $B\mathbb{N}$ depicted as:



Given two morphisms $\bullet \xrightarrow{n} \bullet$ and $\bullet \xrightarrow{m} \bullet$, then their composition is the morphism $\bullet \xrightarrow{n+m} \bullet$. Notice the difference with the category of Example 1.2.22 when $\mathbb{N}$ was regarded as a poset.

Exercise 1.2.34. Given a directed graph, show how to define a category in which you freely add all possible compositions of its edges and identities on vertices? What is the category obtained from the directed graph with a single loop?

1.2.3 Groupoids

Since a group G is a monoid in which every element is a unit, we see that the associated category with one object BG, as in Example 1.2.28, is a category with one object in which every morphism is an isomorphism.

Definition 1.2.35. A *groupoid* is a category in which every morphism is an isomorphism.

Example 1.2.36. A groupoid with one object defines precisely a group. Conversely, given a group G, it defines uniquely a groupoid BG.

Example 1.2.37. Define $\mathbb{I}$ to be the category with two objects and exactly one morphism from one object to another. It can be depicted as (we omitted the identity morphisms):

$$\begin{tikzcd} \bullet \arrow[r, bend left] & \bullet \arrow[l, bend left] \end{tikzcd}$$

Since the composition of the maps depicted above must be a map with a domain equalling its codomain, then it must be the identity by unicity. Hence $\mathbb{I}$ is a groupoid.

Exercise 1.2.38. Let R be a ring. Define $\mathsf{GL}(R)$ to be the following category.

1. $\mathrm{Ob}(\mathsf{GL}(R)) = \{1, 2, 3, \ldots\}$.
2. $\mathrm{Hom}_{\mathsf{GL}(R)}(n, m) = \begin{cases} \mathrm{GL}_n(R) & \text{if } n = m \\ \emptyset & \text{otherwise.} \end{cases}$
3. Composition induced by product of matrices.

Verify that $\mathsf{GL}(R)$ is a groupoid.

Definition 1.2.39. A category $\mathcal{C}$ is *discrete* if the only morphisms are the identities. In other words, for all objects X and Y in $\mathcal{C}$, we have:

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) = \begin{cases} \{\mathrm{id}_X\} & \text{if } X = Y \\ \emptyset & \text{otherwise.} \end{cases}$$

Example 1.2.40. Any discrete category is a groupoid.

Example 1.2.41. Given a set $\mathcal{S}$ (or more generally a class), define $\mathcal{S}_{\mathrm{disc}}$ to be the discrete category with objects $\mathcal{S}$. Given a category $\mathcal{C}$, denote $\mathcal{C}_{\mathrm{disc}}$ the discrete category $\mathrm{Ob}(\mathcal{C})_{\mathrm{disc}}$.

Definition 1.2.42. Given a category $\mathcal{C}$, its *maximal groupoid* $\mathcal{C}^{\cong}$, is the category defined as follows.

1. Define $\mathrm{Ob}(\mathcal{C}^{\cong}) = \mathrm{Ob}(\mathcal{C})$.
2. Given objects X and Y, the set $\mathrm{Hom}_{\mathcal{C}^{\cong}}(X, Y) \subseteq \mathrm{Hom}_{\mathcal{C}}(X, Y)$ is defined to be the set of all isomorphisms from X to Y.
3. Since the composition of isomorphisms is an isomorphism, the composition $\mathrm{Hom}_{\mathcal{C}}(Y, Z) \times \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$ restricts to a composition $\mathrm{Hom}_{\mathcal{C}^{\cong}}(Y, Z) \times \mathrm{Hom}_{\mathcal{C}^{\cong}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}^{\cong}}(X, Z)$.

We shall see in Exercise 1.2.53 that $\mathcal{C}^{\cong}$ is the largest groupoid contained in $\mathcal{C}$, hence the name "maximal".

Exercise 1.2.43. Let M be a monoid. Recall we can regard it as a category BM with one object. Recall also that we denote by $M^{\times}$ its units. Show that $(BM)^{\cong} = B(M^{\times})$

Exercise 1.2.44. Let R be a ring. Define $\mathsf{Mat}(R)$ to be the following category.

- $\mathrm{Ob}(\mathsf{Mat}(R)) = \mathbb{N}$.
- $\mathrm{Hom}_{\mathsf{Mat}(R)}(n, m) = \mathcal{M}_{m \times n}(R)$, the set of matrices with m-rows and n-columns.
- Composition is induced by matrix multiplication: given matrices $A : n \to m$ and $B : m \to k$ then $B \circ A := BA \in \mathcal{M}_{k \times n}(R)$.

Verify that $\mathsf{Mat}(R)$ is a category and its maximal groupoid is $\mathsf{GL}(R)$.

Exercise 1.2.45. What is the maximal groupoid of the categories obtained in Exercise 1.2.31?

Exercise 1.2.46. Explain why $\mathcal{C}^{\cong}$ can never equal $\mathbb{0}$ (unless $\mathcal{C} = \mathbb{0}$).

1.2.4 Subcategories

We have seen that from a category $\mathcal{C}$, we could restrict either its class of objects, or its sets of morphisms, or both. If one regards a category as a monoid with many objects, since there are submonoids, this leads to the notion of a subcategory.

Definition 1.2.47. Given a category $\mathcal{C}$, a *subcategory* $\mathcal{D}$ consists of:

1. a subcollection $\mathrm{Ob}(\mathcal{D}) \subseteq \mathrm{Ob}(\mathcal{C})$
2. for each X and Y in $\mathcal{D}$, a subset $\mathrm{Hom}_{\mathcal{D}}(X,Y) \subseteq \mathrm{Hom}_{\mathcal{C}}(X,Y)$

such that $\mathcal{D}$ becomes itself a category with the composition induced from $\mathcal{C}$, i.e.:

- if $X \in \mathcal{D}$, then id_X is a morphism $\mathcal{D}$;
- if $f : X \to Y$ is a morphism in $\mathcal{D}$, then X and Y are objects in $\mathcal{D}$;
- if $f : X \to Y$ and $g : Y \to Z$ are morphisms in $\mathcal{D}$, then $g \circ f : X \to Z$ is a morphism in $\mathcal{D}$.

A subcategory $\mathcal{D}$ of $\mathcal{C}$ is *full* if $\mathrm{Hom}_{\mathcal{D}}(X,Y) = \mathrm{Hom}_{\mathcal{C}}(X,Y)$ for all $X, Y \in \mathcal{D}$.

Example 1.2.48. The category Grp is a subcategory of Set but it is not full: not ever set map between groups is a homomorphism. Similarly, we get that Ring is a non-full subcategory of Ab, Mon is a non-full subcategory of Set etc.

Example 1.2.49. The category Ab of Abelian groups is a full subcategory of Grp.

Example 1.2.50. We can define a category $\mathsf{Set}_{\mathrm{inj}}$ with same objects as in Set, but the morphisms are only injective functions of sets. Composition is the same as the composite of injective functions is an injective function and the identity function is always injective. An isomorphism is precisely an injective function that is surjective, i.e. a bijection. In many aspects, the category $\mathsf{Set}_{\mathrm{inj}}$ is similar to Set, but we shall see that these categories are not the same. The category $\mathsf{Set}_{\mathrm{inj}}$ is a subcategory that is not full.

Exercise 1.2.51. Let M be a monoid. Let $N \subseteq M$ be a subset. Show that N is a submonoid if and only if BN is a subcategory of BM. Conclude that a subset H of a group G is a subgroup if and only if BH is a groupoid and a subcategory of the groupoid BG.

Example 1.2.52. Given a category $\mathcal{C}$, then $\mathcal{C}_{\mathrm{disc}}$ and $\mathcal{C}^{\cong}$ are subcategories of $\mathcal{C}$ that are not full.

Exercise 1.2.53. Given a category $\mathcal{C}$, show that if $\mathcal{D}$ is a subcategory $\mathcal{C}$ and a groupoid, then $\mathcal{D}$ is a subcategory of the maximal groupoid $\mathcal{C}^{\cong}$.

Exercise 1.2.54. Show that given a category $\mathcal{C}$, a subclass of objects in $\mathcal{C}$ defines uniquely a full subcategory of $\mathcal{C}$. Apply this to define the category CMon of commutative monoids as a full subcategory of Mon.

1.3 Construction on categories

Given a category, we can construct new categories.

1.3.1 Product of categories

Definition 1.3.1. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Define their *Cartesian product* $\mathcal{C} \times \mathcal{D}$ to be the following category.

1. Its class of objects is $\mathrm{Ob}(\mathcal{C} \times \mathcal{D}) = \mathrm{Ob}(\mathcal{C}) \times \mathrm{Ob}(\mathcal{D})$.

2. Given objects $C, C' \in \mathcal{C}$ and $D, D' \in \mathcal{D}$, define

$$\mathrm{Hom}_{\mathcal{C}\times\mathcal{D}}\Big((C,D),(C',D')\Big) = \mathrm{Hom}_{\mathcal{C}}(C,C') \times \mathrm{Hom}_{\mathcal{D}}(D,D')$$

   In other terms, a morphism $(C,D) \to (C',D')$ is denoted (f,g) and consists of a morphism $f : C \to C'$ in $\mathcal{C}$ and a morphism $g : D \to D'$ in $\mathcal{D}$.

3. Composition is induced by the compositions in $\mathcal{C}$ and $\mathcal{D}$. Given morphisms $(C,D) \overset{(f,g)}{\to} (C',D')$ and $(C',D') \overset{(f',g')}{\to} (C'',D'')$ in $\mathcal{C}\times\mathcal{D}$, define $(g',f') \circ (g,f)$ to be $(g' \circ g, f' \circ f) : (C,D) \to (C'',D'')$.

Exercise 1.3.2. Verify that $\mathcal{C}\times\mathcal{D}$ is indeed a category.

Example 1.3.3. The product $\mathbb{2}\times\mathbb{2}$ can be depicted as:

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

Exercise 1.3.4. Let G and H be monoids or groups. Show that $B(G\times H)$ can be regarded as $BG \times BH$.

1.3.2 Slice categories

Definition 1.3.5. Let $\mathcal{C}$ be a category. Fix C an object in $\mathcal{C}$. The *slice category of* $\mathcal{C}$ *over* C, denoted $\mathcal{C}_{/C}$, is defined as follows.

1. Its class of objects $\mathrm{Ob}(\mathcal{C}_{/C})$ consists of pairs (X,f) in which $C \in \mathcal{C}$ and $f : X \to C$ is a morphism in $\mathcal{C}$.

2. Given objects (X,f) and (Y,g) in $\mathcal{C}_{/C}$, a morphism $\alpha : (X,f) \to (Y,g)$ consists of a morphism $\alpha : X \to Y$ in $\mathcal{C}$ such that $g \circ \alpha = f$, i.e. the following diagram commutes in $\mathcal{C}$:

$$\begin{tikzcd} X \arrow[rr, "\alpha"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & C. & \end{tikzcd}$$

3. Composition in $\mathcal{C}_{/C}$ is determined by composition in $\mathcal{C}$. Given morphisms $\alpha : (X,f) \to (Y,g)$ and $\beta : (Y,g) \to (Z,h)$ in $\mathcal{C}_{/C}$, then the composition $\beta\circ\alpha : X \to Z$ remains over C:

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[dr, "f"'] \arrow[rr, bend left, "\beta\circ\alpha"] & Y \arrow[r, "\beta"] \arrow[d, "g"] & Z \arrow[dl, "h"] \\ & C. & \end{tikzcd}$$

We can construct a similar definition and be under an object C instead of over.

Definition 1.3.6. Let $\mathcal{C}$ be a category. Fix C an object in $\mathcal{C}$. The *slice category of* $\mathcal{C}$ *under* C, denoted $\mathcal{C}^{\backslash C}$, is defined as follows.

1. Its class of objects $\mathrm{Ob}(\mathcal{C}^{\backslash C})$ consists of pairs (X,f) in which $C \in \mathcal{C}$ and $f : C \to X$ is a morphism in $\mathcal{C}$.

2. Given objects (X, f) and (Y, g) in $\mathcal{C}^{\backslash C}$, a morphism $\alpha : (X, f) \to (Y, g)$ consists of a morphism $\alpha : X \to Y$ in $\mathcal{C}$ such that $\alpha \circ f = g$, i.e. the following diagram commutes in $\mathcal{C}$:

$$\begin{tikzcd} & C \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X \arrow[rr, "\alpha"'] & & Y. \end{tikzcd}$$

3. Composition in $\mathcal{C}^{\backslash C}$ is determined by composition in $\mathcal{C}$. Given morphisms $\alpha : (X, f) \to (Y, g)$ and $\beta : (Y, g) \to (Z, h)$ in $\mathcal{C}^{\backslash C}$, then the composition $\beta \circ \alpha : X \to Z$ remains under C:

$$\begin{tikzcd} & C \arrow[dl, "f"'] \arrow[d, "g"] \arrow[dr, "h"] & \\ X \arrow[r, "\alpha"'] \arrow[rr, bend right, "\beta\circ\alpha"'] & Y \arrow[r, "\beta"'] & Z. \end{tikzcd}$$

Example 1.3.7. Consider the category Set. Denote $\{*\}$ a singleton. We often write $\mathsf{Set}^{\backslash\{*\}}$ by Set_* and refer to it as the *category of pointed sets*. A function $\{*\} \to X$ is picking up an element $x_0 \in X$. Therefore an object in Set_* consists of a pair (X, x_0) in which X is a set and x_0 is a fixed element in X. A morphism $(X, x_0) \to (Y, y_0)$ in Set_* is determined by a function $f : X \to Y$ such that $f(x_0) = y_0$.

Exercise 1.3.8. Did the choice of the singleton $\{*\}$ mattered in the example above?

Example 1.3.9. Given any set X, there is a unique function $X \to \{*\}$. Therefore we see that $\mathsf{Set}_{/\{*\}}$ is similar to Set.

1.3.3 Skeleton

Example 1.3.10. Given any category $\mathcal{C}$, we see that isomorphisms define an equivalence relation on objects. We can therefore define a category $\mathcal{C}/\cong$ for which objects are $\mathrm{Ob}(\mathcal{C})/\cong$ and $\mathrm{Hom}_{\mathcal{C}/\cong}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(X, Y)/\cong$. We see that $\mathcal{C}/\cong$ is not isomorphic to $\mathcal{C}$ but for a category theorist they should be.

1.3.4 Duality principle

Definition 1.3.11. Given a category $\mathcal{C}$, its *opposite category*, denoted $\mathcal{C}^{\mathsf{op}}$, is a category with same object as in $\mathcal{C}$ in which we abstractly reverse the directions of the arrows. Formally:

1. Its class of objects is $\mathrm{Ob}(\mathcal{C}^{\mathsf{op}}) = \mathrm{Ob}(\mathcal{C})$.

2. Given objects X and Y, define $\mathrm{Hom}_{\mathcal{C}^{\mathsf{op}}}(X, Y) := \mathrm{Hom}_{\mathcal{C}}(Y, X)$. We rewrite a map $f : Y \to X$ in $\mathcal{C}$ by $f^{\mathsf{op}} : X \to Y$ in $\mathcal{C}^{\mathsf{op}}$.

3. Given objects X, Y and Z, define:

$$\begin{array}{rcl} \mathrm{Hom}_{\mathcal{C}^{\mathsf{op}}}(Y, Z) \times \mathrm{Hom}_{\mathcal{C}^{\mathsf{op}}}(X, Y) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}^{\mathsf{op}}}(X, Z) \\ (g^{\mathsf{op}}, f^{\mathsf{op}}) & \longmapsto & g^{\mathsf{op}} \circ_{\mathsf{op}} f^{\mathsf{op}} := (f \circ g)^{\mathsf{op}}. \end{array}$$

Exercise 1.3.12. Check that $\mathcal{C}^{\mathsf{op}}$ defines indeed a category.

Exercise 1.3.13. Prove that $(\mathcal{C}^{\mathsf{op}})^{\mathsf{op}} = \mathcal{C}$.

Exercise 1.3.14. Recall that if M is a monoid, we can define M^{op} is opposite monoid. Show that $B(M^{\mathsf{op}}) = (BM)^{\mathsf{op}}$.

Example 1.3.15. If $\mathbb{P}$ is a poset, we can obtain $\mathbb{P}^{\mathsf{op}}$ by reversing the order. For instance, as in Example 1.2.22, if we view the poset $(\mathbb{N}, \leq)$ as a category $\mathbb{N}$:

$$\bullet \longrightarrow \bullet \longrightarrow \cdots \longrightarrow \bullet \longrightarrow \cdots,$$

then its opposite category $\mathbb{N}^{\mathsf{op}}$ can be depicted as:

$$\bullet \longleftarrow \bullet \longleftarrow \cdots \longleftarrow \bullet \longleftarrow \cdots.$$

Essentially the above category is the one associated to the poset $(\mathbb{N}, \geq)$.

Exercise 1.3.16. Let $\mathcal{C}$ be a category and let C be a fixed object in $\mathcal{C}$. Show that $\left(\mathcal{C}_{/C}\right)^{\mathsf{op}} = \mathcal{C}^{\backslash C}$.

1.4 Functors

Since categories are mathematical structures, there must be a notion of morphisms between them. If we regard categories as monoids with many objects, these morphisms should be thought as a generalization of homomorphisms of monoids. These morphisms go from one category to another. Herein lies the core motivation of category theory: we can travel between the world of groups, or rings, or topological spaces etc, and observe the interactions between these mathematical worlds. A morphism between categories is called a functor.

Definition 1.4.1. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *functor F from $\mathcal{C}$ to $\mathcal{D}$*, denoted $F : \mathcal{C} \to \mathcal{D}$, is the following data.

1. A function on the objects, also denoted F:

$$\begin{array}{rcl} \mathrm{Ob}(\mathcal{C}) & \longrightarrow & \mathrm{Ob}(\mathcal{D}) \\ X & \longmapsto & F(X). \end{array}$$

2. For each pair of objects $X, Y \in \mathcal{C}$, a function on the hom-sets, also denoted F:

$$\begin{array}{rcl} \mathrm{Hom}_{\mathcal{C}}(X, Y) & \longrightarrow & \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y)) \\ \left(X \xrightarrow{f} Y\right) & \longmapsto & \left(F(X) \xrightarrow{F(f)} F(Y)\right) \end{array}$$

This data must follow the following two axioms.

Composition preserving Denote $\circ_{\mathcal{C}}$ and $\circ_{\mathcal{D}}$ the compositions of morphisms in $\mathcal{C}$ and $\mathcal{D}$ respectively. Given $f : X \to Y$ and $g : Y \to Z$ in $\mathcal{C}$, we have the following equality of morphisms in $\mathcal{D}$:

$$F(g \circ_{\mathcal{C}} f) = F(g) \circ_{\mathcal{D}} F(f).$$

Identities preserving For any object $X \in \mathcal{C}$, we have the equality of morphisms in $\mathcal{D}$:

$$F(\mathrm{id}_X) = \mathrm{id}_{F(X)}.$$

Subsequently, when defining a functor $F : \mathcal{C} \to \mathcal{D}$ we shall specify the functions on objects and morphisms at the same time as follows:

$$\begin{array}{rcl} \mathcal{C} & \longrightarrow & \mathcal{D} \\ X & \longmapsto & F(X) \\ \left(X \xrightarrow{f} Y\right) & \longmapsto & \left(F(X) \xrightarrow{F(f)} F(Y)\right). \end{array}$$

Once we specify the two functions, we need to verify it is composition and identities preserving.

Example 1.4.2. Let $\mathcal{C}$ be a category. Define the identity functor $\mathrm{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ as:

$$\begin{aligned} \mathrm{id}_{\mathcal{C}} : \mathcal{C} &\longrightarrow \mathcal{C} \\ X &\longmapsto X \\ \left(X \xrightarrow{f} Y\right) &\longmapsto \left(X \xrightarrow{f} Y\right). \end{aligned}$$

We see immediately that $\mathrm{id}_{\mathcal{C}}(g \circ f) = g \circ f = \mathrm{id}_{\mathcal{C}}(g) \circ \mathrm{id}_{\mathcal{C}}(f)$, for any composable morphisms f and g, and $\mathrm{id}_{\mathcal{C}}(\mathrm{id}_X) = \mathrm{id}_X = \mathrm{id}_{\mathrm{id}_{\mathcal{C}}(X)}$, for all object X in $\mathcal{C}$.

Example 1.4.3. Recall from Example 1.2.28 that a monoid can be regarded as a category with one object. Let $f : M \to N$ be a homomorphism of monoids. This defines a functor $Bf : BM \to BN$:

$$\begin{aligned} Bf : BM &\longrightarrow BN \\ \star &\longmapsto \star \\ \left(\star \xrightarrow{m} \star\right) &\longmapsto \left(\star \xrightarrow{f(m)} \star\right). \end{aligned}$$

Let us verify it preserves composition and identities. Given $m, n \in M$, since f is a homomorphism, we have $f(mn) = f(m)f(n)$. This precisely translates to $Bf(m \circ n) = Bf(m) \circ Bf(n)$. Recall that $\mathrm{id}_\star$ in BM is the neutral element e_M of M. Since f is a homomorphism, then $f(e_M) = e_N$, the neutral element of N. Thus $Bf(\mathrm{id}_\star) = \mathrm{id}_\star$. Therefore, Bf is indeed a functor.

Exercise 1.4.4. Suppose M and N are monoids and let $F : BM \to BN$ be any functor. Show that there is a unique homomorphism of monoids $f : M \to N$ such that $Bf = F$. Conclude there is a correspondence between homomorphisms of monoids and functors between categories with one object.

Example 1.4.5. If G and H are groups, then a group homomorphism $f : G \to H$ defines a functor $Bf : BG \to BH$. Moreover, any functor $F : BG \to BH$ defines a group homomorphisms $f : G \to H$ such that $F = Bf$.

Example 1.4.6. A mathematical construction can be viewed as functorial in several ways. Let R be a ring. Recall that we can define the polynomial ring $R[x_1, \ldots, x_n]$ with n-variables. This is functorial if we vary the ring: i.e. there is a functor:

$$\begin{aligned} \mathsf{Ring} &\longrightarrow \mathsf{Ring} \\ R &\longmapsto R[x_1, \ldots, x_n] \\ \left(R \xrightarrow{f} S\right) &\longmapsto \begin{pmatrix} R[x_1, \ldots, x_n] \xrightarrow{f} S[x_1, \ldots, x_n] \\ \displaystyle\sum_{i,j \geq 0}^{n} a_{ij} x_i^j \mapsto \sum_{i,j \geq 0}^{n} f(a_{ij}) x_i^j \end{pmatrix} \end{aligned}$$

On the other hand, we could have also varied the amount of variables. If we denote Fin the full subcategory of Set spanned by finite sets, then there is a functor:

$$\begin{aligned} \mathrm{Fin} &\longrightarrow \mathsf{Ring} \\ \{1, \ldots, n\} &\longmapsto R[x_1, \ldots, x_n] \\ \left(\{1, \ldots, n\} \xrightarrow{\sigma} \{1, \ldots, m\}\right) &\longmapsto \begin{pmatrix} R[x_1, \ldots, x_n] \xrightarrow{\sigma} R[x_1, \ldots, x_m] \\ \displaystyle\sum_{i,j \geq 0}^{n} a_{ij} x_i^j \mapsto \sum_{i,j \geq 0}^{n} a_{\sigma(i)j} x_{\sigma(i)}^j \end{pmatrix} \end{aligned}$$

We can record this fact by capturing the two variances into one functor:

$$\begin{aligned}\mathsf{Ring} \times \mathrm{Fin} &\longrightarrow \mathsf{Ring} \\ (R, \{1, \dots, n\}) &\longmapsto R[x_1, \dots, x_n] \\ (f, \sigma) &\longmapsto \begin{pmatrix} R[x_1, \dots, x_n] \overset{(f,\sigma)}{\to} S[x_1, \dots, x_m] \\ \displaystyle\sum_{i,j\geq 0}^{n} a_{ij} x_i^j \mapsto \sum_{i,j\geq 0}^{n} f(a_{\sigma(i)j}) x_{\sigma(i)}^j \end{pmatrix}\end{aligned}$$

Exercise 1.4.7. Let R be a ring. Recall that for a (possibly infinite) set, we can define $R[X]$. Show this leads to functors $\mathsf{Set} \to \mathsf{Ring}$ and $\mathsf{Ring} \times \mathsf{Set} \to \mathsf{Ring}$.

Exercise 1.4.8. Let $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ be categories. Show that the data of a functor $\mathcal{E} \to \mathcal{C} \times \mathcal{D}$ is equivalent to the data of two functors $\mathcal{E} \to \mathcal{C}$ and $\mathcal{E} \to \mathcal{D}$.

Example 1.4.9. Recall that for a ring R we denote by $R^\times$ its group of units. If $f : R \to S$ is a ring homomorphism, then if $r \in R$ is a unit, then so is $f(r)$. Thus we can restrict and corestrict $f^\times : R^\times \to S^\times$. This defines a functor:

$$\begin{aligned}(-)^\times : \mathsf{Ring} &\longrightarrow \mathsf{Grp} \\ R &\longmapsto R^\times \\ \left(R \overset{f}{\to} S\right) &\longmapsto \left(R^\times \overset{f^\times}{\to} S^\times\right)\end{aligned}$$

Example 1.4.10. A lot of mathematical structures are sets with additional structures. For instance, a monoid $(M, *, e_M)$ is a set M with the extra data of a multiplication and unity. Moreover, any monoid homomorphism $M \to N$ is a set map with the extra requirement that it must preserves identity and multiplication. Forgetting this data defines a functor:

$$\begin{aligned}\mathsf{Mon} &\longrightarrow \mathsf{Set} \\ (M, *, e_M) &\longmapsto M \\ \left((M, *, e_m) \overset{f}{\to} (N, \odot, e_N)\right) &\longmapsto \left(M \overset{f}{\to} N\right)\end{aligned}$$

Such functor is called a *forgetful functor*, or *underlying functor*, and is often denoted U. It occurs in many instances: a ring is an Abelian group with a multiplication, an Abelian group is a group in which the multiplication is commutative, a group is a monoid for which every element has an inverse, and we just saw that a monoid is a set with extra structure. So we have all these forgetful functors:

$$\mathsf{Ring} \to \mathsf{Ab} \to \mathsf{Grp} \to \mathsf{Mon} \to \mathsf{Set}.$$

Generally they are all denoted U and there is usually no ambiguity. In practice, we often omit U. For instance we might prefer to say "the set M", or "M regarded as a set" instead of writing $U(M)$, for M a monoid and $U : \mathsf{Mon} \to \mathsf{Set}$.

Example 1.4.11. Given a monoid $(M, *, e_M)$ we can forget the multiplication but still keep track of the choice of unity. The assignment $(M, *, e_M) \to (M, e_M)$ defines a forgetful functor $\mathsf{Mon} \to \mathsf{Set}_*$.

Example 1.4.12. Given a set X, one can define the free group on X, denoted $F(X)$ formed of all possible words in X together with concatenation as multiplication and the empty word as unity. The definition can be extended to map of sets and thus we obtain a functor $F : \mathsf{Set} \to \mathsf{Grp}$. There exists a nice connection between F and the forgetful U from previous example: given any set X and group G, there is an isomorphism of sets (i.e. bijection):

$$\mathrm{Hom}_{\mathsf{Grp}}(F(X), G) \cong \mathrm{Hom}_{\mathsf{Set}}(X, U(G)).$$

This is precisely a reformulation of the universal property of free groups. In fact, given a forgetful functor $U : \mathcal{C} \to \mathsf{Set}$, the free functor $F : \mathsf{Set} \to \mathcal{C}$ will be defined as the unique functor such that we obtain a "univesal" isomorphism:

$$\mathrm{Hom}_{\mathcal{C}}(F(X), C) \cong \mathrm{Hom}_{\mathsf{Set}}(X, U(C)),$$

given any set X and object $C \in \mathcal{C}$. We will make this idea precise in the next sections, it is important to notice now that there seems to be a general pattern with free objects and their universal properties.

Exercise 1.4.13. Let $U : \mathsf{Set}_* \to \mathsf{Set}$ be the forgetful functor on pointed sets. Given a set X, denote $X_+ = (X \coprod \{*\}, *)$ the set where we have added a point $*$ to X and chose it as a basepoint. Given a set map $f : X \to Y$, extend it to a set map $f_+ : X_+ \to Y_+$ by $f(*) = *$. Show this defines a functor $(-)_+ : \mathsf{Set} \to \mathsf{Set}_*$ such that we obtain an isomorphism of sets (i.e. bijection):

$$\mathrm{Hom}_{\mathsf{Set}_*}(X_+, (Y, y_0)) \cong \mathrm{Hom}_{\mathsf{Set}}(X, Y),$$

for any set X and pointed set (Y, y_0), where we denoted $Y = U(Y, y_0)$.

Example 1.4.14. Recall that for any set S, there are unique set maps $\emptyset \to S$ and $S \to \{*\}$. In a similar fashion, given any category $\mathcal{C}$, there are unique functors $\mathbb{0} \to \mathcal{C}$ and $\mathcal{C} \to \mathbb{1}$.

Example 1.4.15. Recall that choosing an element x in a set S defines a map $x : \{*\} \to S$. Similarly, choosing an object X in a category $\mathcal{C}$ amounts precisely to a functor $X : \mathbb{1} \to \mathcal{C}$.

Exercise 1.4.16. Show that choosing a morphism $f : X \to Y$ in $\mathcal{C}$ amounts precisely to a functor $f : \mathbb{2} \to \mathcal{C}$. Show that choosing an isomorphism $f : X \to Y$ in $\mathcal{C}$ amounts precisely to a functor $f : \mathbb{I} \to \mathcal{C}$.

One key importance of functors is that they detect non-isomorphic objects. Therefore given a functor $F : \mathcal{C} \to \mathcal{D}$, if you want to determine if objects in $\mathcal{C}$ are not isomorphic, you can use the functor F to travel in a different realm.

Theorem 1.4.17. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Suppose $f : X \to Y$ is an isomorphism in $\mathcal{C}$, then $F(f) : F(X) \to F(Y)$ is an isomorphism in $\mathcal{D}$ and $F(f)^{-1} = F(f^{-1})$. In particular, a functor *preserves* isomorphisms: if $X \cong Y$ in $\mathcal{C}$, then $F(X) \cong F(Y)$ in $\mathcal{D}$.

Proof. Since $f : X \to Y$ is an isomorphism in $\mathcal{C}$, there exists $f^{-1} : Y \to X$ in $\mathcal{C}$ such that $f^{-1} \circ f = \mathrm{id}_X$ and $f \circ f^{-1} = \mathrm{id}_Y$. Therefore:

$$\begin{aligned} F(f^{-1}) \circ F(f) &= F(f^{-1} \circ f), \text{ by composition preserving,} \\ &= F(\mathrm{id}_X) \\ &= \mathrm{id}_{F(X)}, \text{ by identities preserving.} \end{aligned}$$

Similarly, we obtain $F(f) \circ F(f^{-1}) = \mathrm{id}_{F(Y)}$. Thus $F(f) : F(X) \to F(Y)$ is an isomorphism in $\mathcal{D}$ with inverse $F(f^{-1}) : F(Y) \to F(X)$. □

Corollary 1.4.18. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Let $F : \mathcal{C} \to \mathcal{D}$ be functors. Let X and Y be objects in $\mathcal{C}$. If $F(X) \ncong F(Y)$ in $\mathcal{D}$, then $X \ncong Y$ in $\mathcal{C}$.

Example 1.4.19. Two groups cannot be isomorphic if their cardinals were not equal. This follows from the forgetful functor $\mathsf{Ring} \to \mathsf{Set}$.

Example 1.4.20. The ring $\mathbb{Z}$ and $\mathbb{Q}$ cannot be isomorphic (even though they have the same cardinality) because $\mathbb{Z}^\times \cong C_2 \ncong \mathbb{Q} - \{0\} = \mathbb{Q}^\times$.

Warning 1.4.21. In general, given a functor $F : \mathcal{C} \to \mathcal{D}$, if $F(X) \cong F(Y)$ in $\mathcal{D}$, there is no reason to expect that $X \cong Y$ in $\mathcal{C}$.

Example 1.4.22. Given any category $\mathcal{C}$, the unique functor $F : \mathcal{C} \to \mathbb{1}$ forces that $F(X) \cong F(Y)$ for any two object X and Y in $\mathcal{C}$.

Example 1.4.23. Consider the functor $(-)^\times : \mathsf{Ring} \to \mathsf{Grp}$ and the rings $\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z}$. They are not isomorphic but $\mathbb{Z}^\times \cong C_2 \cong (\mathbb{Z}/3\mathbb{Z})^\times$.

Definition 1.4.24. A functor $F : \mathcal{C} \to \mathcal{D}$ is said to be *conservative* if it *reflects* isomorphisms: if $f : X \to Y$ is in $\mathcal{C}$ such that $F(f) : F(X) \to F(Y)$ is an isomorphism in $\mathcal{D}$, then f is an isomorphism in $\mathcal{C}$.

Example 1.4.25. The forgetful functor $\mathsf{Grp} \to \mathsf{Set}$ is conservative: a homomorphism of groups that is a bijection is an isomorphism. Many of the forgetful functors we have seen are conservative. However, not all forgetful functors are conservative (example in topology).

Example 1.4.26. Even though the forgetful functor $\mathsf{Ring} \to \mathsf{Grp}$ is conservative, the functor $(-)^\times : \mathsf{Ring} \to \mathsf{Grp}$ is not. For instance, consider the quotient map $\gamma : \mathbb{Z} \to \mathbb{Z}/3\mathbb{Z}$. Then $\gamma^\times : \mathbb{Z}^\times \to (\mathbb{Z}/3\mathbb{Z})^\times$ is an isomorphism but γ is not.

Example 1.4.27. Given a functor $F : \mathcal{C} \to \mathcal{D}$ and a functor $G : \mathcal{D} \to \mathcal{E}$, then just as we can compose functions or more generally morphisms in a category, we can compose functors and define $G \circ F : \mathcal{C} \to \mathcal{E}$ to be defined as follows:

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & \mathcal{E} \\ C & \longmapsto & G(F(C)) \\ \left(C \xrightarrow{f} C'\right) & \longmapsto & \left(G(F(C)) \xrightarrow{G(F(f))} G(F(C'))\right). \end{array}$$

Exercise 1.4.28. Verify that $G \circ F$ of Example 1.4.27 is indeed a functor.

It is tempting to consider a category of categories in which objects are categories and morphisms are functors. Unfortunately we encounter size issues in the same way that there cannot be a set of all sets. To palliate this issue we can consider a large category of all categories. A large category is not a category with our terminology as its hom can be a class.

Definition 1.4.29. Define CAT the large category of all categories as follows.

1. Its objects are categories.

2. Given categories $\mathcal{C}$ and $\mathcal{D}$, the class $\mathrm{Hom}_{\mathsf{CAT}}(\mathcal{C}, \mathcal{D})$ comprises of all functors $\mathcal{C} \to \mathcal{D}$.

3. Composition of functors is defined as in Example 1.4.27.

One can check this structure gives indeed a large category, in which the identities are given as in Example 1.4.2.

We shall see soon that the class $\mathrm{Hom}_{\mathsf{CAT}}(\mathcal{C}, \mathcal{D})$ can itself be endowed with a category structure that we will denote $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$.

To avoid size issues, we may consider instead a category of small categories.

Definition 1.4.30. Define Cat as the full subcategory of CAT spanned by small categories. In this instance, the morphisms assemble to a set and not a class.

Example 1.4.31. We have seen that for any monoid M we can associate a category BM with one object, this defines a functor $B : \mathsf{Mon} \to \mathsf{Cat}$.

Exercise 1.4.32. Define Grpd as the full subcategory of Cat spanned by groupoids. Show we obtain a functor $B : \mathsf{Grp} \to \mathsf{Grpd}$.

1.5 Embedding categories

Definition 1.5.1. A functor $F : \mathcal{C} \to \mathcal{D}$ is said to be *faithful* if for all objects X and Y in $\mathcal{C}$, the set map $F : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$ is injective. In other words, if $f : X \to Y$ and $g : X \to Y$ are maps in $\mathcal{C}$, then if $F(f) = F(g)$ as maps $F(X) \to F(Y)$ in $\mathcal{D}$, then $f = g$.

Example 1.5.2. Typically, forgetful functors are faithful functors. For instance two homomorphisms of groups are equal if they are equal as set maps.

Example 1.5.3. Given a category $\mathcal{C}$, there exists a unique functor $\mathcal{C} \to \mathbb{1}$. This functor is never faithful unless $\mathcal{C}$ is $\mathbb{0}$ or $\mathbb{1}$.

Example 1.5.4. Let $f : M \to N$ be a homomorphism of monoids. Then the induced functor $Bf : BM \to BN$ is faithful if and only if f is injective.

Definition 1.5.5. A *concrete category* $\mathcal{C}$ is a category together with a faithful functor $U : \mathcal{C} \to \mathsf{Set}$.

Example 1.5.6. The categories Mon, Grp, Ab, Set_*, Ring etc are all concrete categories when considering their forgetful functors onto Set.

Exercise 1.5.7. Suppose $F : \mathcal{C} \to \mathcal{D}$ is a faithful functor. Show that if a diagram in $\mathcal{C}$ commutes in $\mathcal{D}$ after applying F, then it commutes in $\mathcal{C}$.

Definition 1.5.8. A functor $F : \mathcal{C} \to \mathcal{D}$ is said to be *full* if for all objects X and Y in $\mathcal{C}$ the set map $F : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}((F(X), F(Y))$ is surjective. In other words, given a map $g : F(X) \to F(Y)$ in $\mathcal{D}$, there exists a map $f : X \to Y$ in $\mathcal{C}$ such that $F(f) = g$.

Example 1.5.9. Let $f : M \to N$ be a homomorphism of monoids. Then the induced functor $Bf : BM \to BN$ is full if and only if f is surjective.

Definition 1.5.10. A functor $F : \mathcal{C} \to \mathcal{D}$ is said to be:

- *essentially injective* if: given objects $X, Y \in \mathcal{C}$, if $F(X) \cong F(Y)$ in $\mathcal{D}$, then $X \cong Y$ in $\mathcal{C}$;
- *essentially surjective* if: for all $D \in \mathcal{D}$, there exists $C \in \mathcal{C}$ such that $F(C) \cong D$ in $\mathcal{D}$.

Warning 1.5.11. It is important to not confuse essentially injective with conservative (Definition 1.4.24). A functor can be essentially injective without being conservative.

Example 1.5.12 (TO DO)**.** Example of a non essentially injective but conservative functor. Example of a conservative functor but non essentially injective.

Proposition 1.5.13. Let $F : \mathcal{C} \to \mathcal{D}$ be a conservative and full functor. Then F is essentially injective.

Proof. Let X and Y be objects in $\mathcal{C}$ and suppose $F(X) \cong F(Y)$ in $\mathcal{D}$. This means there exists a map $g : F(X) \to F(Y)$ in $\mathcal{D}$ that is an isomorphism. Since F is full, there exists a map $f : X \to Y$ in $\mathcal{C}$ such that $F(f) = g$. Since F is conservative, then f must be an isomorphism. Thus $X \cong Y$ in $\mathcal{C}$. □

Example 1.5.14. Choosing an isomorphism $f : X \to Y$ in a category $\mathcal{C}$ defines a functor $f : \mathbb{I} \to \mathcal{C}$ that is conservative and essentially injective but neither full nor faithful in general.

Definition 1.5.15. An *embedding of categories* is a functor $F : \mathcal{C} \to \mathcal{D}$ that is essentially injective and faithful. In this case we say $\mathcal{C}$ is *embedded* in $\mathcal{D}$. If the functor $F : \mathcal{C} \to \mathcal{D}$ is also full, then we say F is a *full embedding of categories* and $\mathcal{C}$ is *fully embedded* in $\mathcal{D}$.

Proposition 1.5.16. Given an embedding of categories $F : \mathcal{C} \to \mathcal{D}$ and objects X, Y in $\mathcal{C}$. Then $X \cong Y$ in $\mathcal{C}$ if and only if $F(X) \cong F(Y)$ in $\mathcal{D}$.

In a non-full embedding of categories $\mathcal{C}$ in $\mathcal{D}$ it is possible that one added more morphisms between objects in $\mathcal{D}$.

Example 1.5.17. A subcategory $\mathcal{C}$ of a category $\mathcal{D}$ defines an embedding of categories $\mathcal{C} \to \mathcal{D}$. A full subcategory $\mathcal{C}$ of $\mathcal{D}$ defines a full embedding of categories. Informally, a (full) embedding of categories $F : \mathcal{C} \to \mathcal{D}$ defines a (full) subcategory in $\mathcal{D}$ comprised of the *essential image* of F: the smallest subcategory of $\mathcal{D}$ containing all objects in $\mathcal{D}$ that are isomorphic to $F(X)$ for some $X \in \mathcal{C}$.

Definition 1.5.18. A functor $F : \mathcal{C} \to \mathcal{D}$ that is both full and faithful is called *fully faithful*.

Proposition 1.5.19. A fully faithful functor is conservative, and thus in particular essentially injective.

Proof. Let $F : \mathcal{C} \to \mathcal{D}$ be a fully faithful functor. Let $f : X \to Y$ be a map in $\mathcal{C}$. Suppose $F(f) : F(X) \to F(Y)$ is an isomorphism in $\mathcal{D}$. This means there exists $g : F(Y) \to F(X)$ in $\mathcal{D}$ such that $g \circ F(f) = \mathrm{id}_{F(X)}$ and $F(f) \circ g = \mathrm{id}_{F(Y)}$. Since F is full, there exists $h : Y \to X$ in $\mathcal{C}$ such that $F(h) = g$. We obtain the equalities:

$$F(h \circ f) = F(h) \circ F(f) = g \circ F(f) = \mathrm{id}_{F(X)} = F(\mathrm{id}_X).$$

As F is faithful, we obtain $h \circ f = \mathrm{id}_X$. We argue similarly to show $f \circ h = \mathrm{id}_Y$. Thus f is an isomorphism in $\mathcal{C}$ with inverse h. Thus F is conservative. □

Corollary 1.5.20. A functor is fully faithful if and only if it is a full embedding of categories.

1.6 Equivalences of categories and natural transformations

In previous section we saw that we can define a morphism between categories called functors. Omitting size issues, categories together with their functors assemble themselves into a category CAT. Therefore this defines a notion of an isomorphism of categories as in Definition 1.2.4. Two categories $\mathcal{C}$ and $\mathcal{D}$ are *isomorphic*, and we write $\mathcal{C} \cong \mathcal{D}$, if there exist functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ such that $G \circ F = \mathrm{id}_{\mathcal{C}}$ and $F \circ G = \mathrm{id}_{\mathcal{D}}$.

However, the definition is often extremely rigid. Indeed, we require that for any object X in $\mathcal{C}$ that $G(F(X))$ is *equal* to X, and given any morphism $f : X \to Y$, we have $G(F(f))$ is equal to f. But in practice, it is more likely that $G(F(X))$ is only isomorphic to X. In that case, we cannot have $G(F(f)) = f$ as they don't have same domains and codomains, but we can ask that they remain compatible. In other words, we would like the following diagram to commute:

$$\begin{tikzcd} G(F(X)) \arrow[r, "\cong"] \arrow[d, "GF(f)"'] & X \arrow[d, "f"] \\ G(F(Y)) \arrow[r, "\cong"] & Y, \end{tikzcd}$$

for any morphism $f : X \to Y$ in $\mathcal{C}$. If the diagrams above was not commutative then if means we have lost information on the morphisms in $\mathcal{C}$.

We make precise the notion here. We begin by two motivating examples.

Example 1.6.1. Let $\mathbb{F}$ be a field. Our first example is motivated from the fact that every finite dimensional $\mathbb{F}$-vector space is isomorphic (but not equal) to $\mathbb{F}^n$ for some $n \geq 0$. First recall we have defined the category $\mathsf{Mat}(\mathbb{F})$ of matrices with coefficients in $\mathbb{F}$. Denote $\mathsf{Vect}_{\mathbb{F}}^{\mathsf{fd}}$ the category for which objects are pairs $(V, \mathcal{B})$ where V is a finite dimensional $\mathbb{F}$-vector space and $\mathcal{B}$ is an ordered (finite) basis of V. Morphisms $(V, \mathcal{B}) \to (V', \mathcal{B}')$ are linear transformations $V \to V'$ with no added requirements. Therefore $\mathsf{Vect}_{\mathbb{F}}^{\mathsf{fd}}$ is a full subcategory of $\mathsf{Vect}_{\mathbb{F}}$. Recall that a matrix $A \in \mathcal{M}_{m\times n}(\mathbb{F})$ can be regarded as a linear transformation:

$$\begin{array}{rcl} A : (\mathbb{F}^n, \mathcal{S}) & \longrightarrow & (\mathbb{F}^m, \mathcal{S}) \\ v & \longmapsto & Av. \end{array}$$

Here we denote $\mathcal{S}$ the standard basis of $\mathbb{F}^n$. This perspective defines a functor:

$$\begin{array}{rcl} \mathsf{Mat}(\mathbb{F}) & \longrightarrow & \mathsf{Vect}_{\mathbb{F}}^{\mathsf{fd}} \\ n & \longmapsto & (\mathbb{F}^n, \mathcal{S}) \\ A \in \mathcal{M}_{m\times n}(\mathbb{F}) & \longmapsto & \Big(A : (\mathbb{F}^n, \mathcal{S}) \to (\mathbb{F}^m, \mathcal{S})\Big). \end{array}$$

Let $\mathcal{B} = (b_1, \dots, b_n)$ be a basis of a vector space V. Then for any $v \in V$, there exist unique scalars $\lambda_1, \dots, \lambda_n \in \mathbb{F}^n$ such that $v = \lambda_1 b_1 + \cdots \lambda_n b_n$. This defines the vector $[v]_{\mathcal{B}} = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix}$ in $\mathbb{F}^n$. Given a linear transformation $T : (V, \mathcal{B}) \to (V', \mathcal{B}')$, we denote the matrix $[T]_{\mathcal{B}}^{\mathcal{B}'}$ comprised of the vectors $[T(b_1)]_{\mathcal{B}'}, \dots, [T(b_n)]_{\mathcal{B}'}$ in its columns. We obtain a functor:

$$\begin{array}{ccc} \mathsf{Vect}_{\mathbb{F}}^{\mathsf{fd}} & \longrightarrow & \mathsf{Mat}(\mathbb{F}) \\ (V, \mathcal{B}) & \longmapsto & \dim(V) \\ \Big((V, \mathcal{B}) \xrightarrow{T} (V', \mathcal{B}')\Big) & \longmapsto & [T]_{\mathcal{B}}^{\mathcal{B}'} \end{array}$$

One one hand the composition:

$$n \longmapsto (\mathbb{F}^n, \mathcal{S}) \longmapsto \dim(\mathbb{F}^n)$$

is the identity on objects and on morphisms. On the other hand, the composition:

$$(V, \mathcal{B}) \longmapsto \dim(V) \longmapsto (\mathbb{F}^{\dim(V)}, \mathcal{S})$$

is in general only an isomorphism given by:

$$\begin{array}{ccc} (V, \mathcal{B}) & \xrightarrow{\cong} & (\mathbb{F}^{\dim(V)}, \mathcal{S}) \\ v & \longmapsto & [v]_{\mathcal{B}}. \end{array}$$

Notice that these isomorphisms are compatible: given any linear transformation $T : (V, \mathcal{B}) \to (V', \mathcal{B}')$, we have the commutative diagram:

$$\begin{tikzcd} (V, \mathcal{B}) \arrow[r, "\cong"] \arrow[d, "T"'] & (\mathbb{F}^{\dim(V)}, \mathcal{S}) \arrow[d, "{[T]_{\mathcal{B}}^{\mathcal{B}'}}"] \\ (V', \mathcal{B}') \arrow[r, "\cong"] & (\mathbb{F}^{\dim(V')}, \mathcal{S}) \end{tikzcd}$$

This expresses the familiar equation $[T(v)]_{\mathcal{B}'} = [T]_{\mathcal{B}}^{\mathcal{B}'} [v]_{\mathcal{B}}$. Therefore, although $\mathsf{Vect}_{\mathbb{F}}^{\mathsf{fd}}$ and $\mathsf{Mat}(\mathbb{F})$ are not isomorphic, they seem to be equivalent in many regards.

Example 1.6.2. From our perspective, a ring is always unital. We shall explain why here.

A ring R is said to be be augmented if there is a ring homomorphism $\varepsilon : R \to \mathbb{Z}$. In fact we can define the category of augmented rings as $\mathsf{Ring}_{/\mathbb{Z}}$. A non-unital ring R is a ring without the axiom of unity, and a non-unital ring homomorphism is a ring that doesn't preserve unity. This defines a category $\mathsf{Ring}_{\circ}$.

Given a non-unital ring R, let $R_+ = R \oplus \mathbb{Z}$. Then one can check that R_+ is a (unital) ring with unity $(0_R, 1)$ and is augmented via the quotient map $R_+ \to R_+/R \cong \mathbb{Z}$. Define a functor:

$$\begin{gathered} (-)_+ : \mathsf{Ring}_{\circ} \longrightarrow \mathsf{Ring}_{/\mathbb{Z}} \\ R \to R_+ \\ \Big(R \xrightarrow{f} S\Big) \longmapsto \Big(R_+ \xrightarrow{f \oplus \mathrm{id}} S_+\Big). \end{gathered}$$

Given an augmented ring (R, ε), we can denote $R_- = \ker(\varepsilon)$, it is a non-unital ring.

$$\begin{gathered} (-)_- : \mathsf{Ring}_{/Z} \longrightarrow \mathsf{Ring}_{\circ} \\ (R, \varepsilon) \longmapsto R_- \\ \Big(R \to S\Big) \longmapsto \Big(R_- \to S_-\Big) \end{gathered}$$

Suppose (R, ε) is an augmented ring. Then $((R, \varepsilon)_-)_+ = \ker(\varepsilon) \oplus \mathbb{Z}$. Since we have an isomorphism:

$$\begin{aligned} \ker(\varepsilon) \oplus \mathbb{Z} &\longrightarrow R \\ (r, 1) &\longmapsto r + 1_R \end{aligned}$$

Similarly, given a non unital ring R, then $(R_+)_- = \ker(R \oplus \mathbb{Z} \to \mathbb{Z}) \cong R$. These are not equal but really isomorphisms. One can view the category of non-unital rings $\mathsf{Ring}_\circ$ as living inside the category of unital rings Ring, as $\mathsf{Ring}_{/\mathbb{Z}}$ is a (non full) subcategory of Ring.

The above example hopefully served as motivation for the need of this coherence data on morphism.

Definition 1.6.3. Let $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{C} \to \mathcal{D}$ be functors. A *natural transformation* from F to G, denoted $\alpha\colon F \Rightarrow G$ is a collection of morphisms $\{\alpha_X\colon F(X) \to G(X) \mid X \in \mathcal{C}\}$ in $\mathcal{D}$ subject to the following requirement. For any morphism $f : X \to Y$ in $\mathcal{C}$, the following diagram commutes in $\mathcal{D}$:

$$\begin{tikzcd} F(X) \arrow[r, "\alpha_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\alpha_Y"'] & G(Y) \end{tikzcd}$$

If $\alpha_X : F(X) \to G(X)$ is an isomorphism in $\mathcal{D}$ for each $X \in \mathcal{C}$, then we say α is a *natural isomorphism* and we write it as $\alpha : F \overset{\sim}{\Rightarrow} G$.

Notation 1.6.4. Instead of saying "let $\alpha : F \Rightarrow G$ be a natural transformation between functors $F, G : \mathcal{C} \to \mathcal{D}$", we may compactly refer to the data as a diagram:

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"{name=U}] \arrow[r, bend right, "G"'{name=D}] & \mathcal{D} \arrow[from=U, to=D, Rightarrow, "\alpha"] \end{tikzcd}$$

TO FINISH...

1.7 Yoneda Lemma

We begin by defining what representable means in the categorical sense. Our result will be stated for the contravariant case. However, all the work in this section can be dualized for covariant functors.

We shall always assume $\mathcal{C}$ to be a *locally small category*, i.e., a category such that for any object C and C' in $\mathcal{C}$, the class of morphisms $\mathcal{C}(C, C')$ is a set. Otherwise we extend Grothendieck universe to extend what we mean by a set.

Let C_0 be a fixed object of $\mathcal{C}$. We define the functor:

$$\begin{aligned} \mathcal{C}(-, C_0)\colon \mathcal{C}^{\mathsf{op}} &\longrightarrow \mathsf{Set} \\ C &\longmapsto \mathcal{C}(C, C_0) \\ C \xrightarrow{f} C' &\longmapsto f^* : \mathcal{C}(C', C_0) \to \mathcal{C}(C, C_0), \end{aligned}$$

where $f^*(\varphi) = \varphi \circ f$, for any φ in $\mathcal{C}(C', C_0)$.

Definition 1.7.1 (Representable Contravariant Functor)**.** Let $\mathcal{C}$ be a locally small category. A functor $F\colon \mathcal{C}^{\mathsf{op}} \to \mathsf{Set}$ is said to be *representable* if there is an object C_0 in $\mathcal{C}$ and a natural isomorphism:

$$e\colon \mathcal{C}(-, C_0) \Rightarrow F.$$

We say that C_0 *represents* F, and C_0 is a *classifying object* for F. Similarly, a functor $F\colon \mathcal{C} \to \mathsf{Set}$ is representable if there is an object C_0 and a natural isomorphism $\mathcal{C}(C_0, -) \Rightarrow F$.

The following lemma, known as the Yoneda Lemma, relates natural transformations $e : \mathcal{C}(-, C_0) \Rightarrow F$ with elements of $F(C_0)$.

Lemma 1.7.2 (Yoneda)**.** Let $F\colon \mathcal{C}^{\mathsf{op}} \to \mathsf{Set}$ be a functor. For any object C_0 in $\mathcal{C}$, there is a one-to-one correspondance between natural transformation $e\colon \mathcal{C}(-, C_0) \Rightarrow F$ and elements u in $F(C_0)$, which is given, for any object C in $\mathcal{C}$, by:

$$\begin{array}{rcl} e_C : \mathcal{C}(C, C_0) & \longrightarrow & F(C) \\ \varphi & \longmapsto & F(\varphi)(u). \end{array}$$

Proof. Suppose we are given a natural transformation $e : \mathcal{C}(-, C_0) \Rightarrow F$. In particular, for any morphism φ in $\mathcal{C}(C, C_0)$, the following diagram commutes:

$$\begin{tikzcd} \mathcal{C}(C_0, C_0) \arrow[r, "e_{C_0}"] \arrow[d, "\varphi^*"'] & F(C_0) \arrow[d, "F(\varphi)"] \\ \mathcal{C}(C, C_0) \arrow[r, "e_C"] & F(C). \end{tikzcd}$$

Evaluating with the identity morphism id_{C_0}, we obtain an element $u = e_{C_0}(\mathrm{id}_{C_0})$ in $F(C_0)$; and commutativity of the previous diagram gives: $e_C(\varphi) = F(\varphi)(u)$.
Conversely, if we are given u in $F(C_0)$, define $e_C : \mathcal{C}(C, C_0) \longrightarrow F(C)$ as before, for all objects C. Naturality follows directly. $\square$

The Yoneda Lemma leads to the following definition.

Definition 1.7.3 (Universal Elements)**.** If $F : \mathcal{C}^{\mathsf{op}} \to \mathsf{Set}$ is a representable functor, given a natural isomorphism $e : \mathcal{C}(-, C_0) \Rightarrow F$, the associated element according to the Yoneda Lemma $u_F := e_{C_0}(\mathrm{id}_{C_0}) \in F(C_0)$ is called the *universal element* of F.

Example 1.7.4. Fix a group homomorphism $f\colon G \to G'$. Given any group H, define H_f to be the set of group homomorphisms $h\colon H \to G$ such that $f \circ h = 0$. Given a group homomorphism $\varphi\colon H \to H'$, we get a set map $\varphi^*\colon H'_f \to H_f$ sending $h\colon H' \to G$ to $h \circ \varphi$. This defines a functor:

$$\begin{aligned} (-)_f \colon \mathsf{Grp}^{\mathsf{op}} &\longrightarrow \mathsf{Set} \\ H &\longmapsto H_f \\ \left(H \xrightarrow{\varphi} H'\right) &\longmapsto \left(H'_f \xrightarrow{f^*} H_f\right). \end{aligned}$$

The universal property of the kernel of f says exactly that this functor is representable by $\ker(f)$ and that its universal element is the inclusion $\ker(f) \hookrightarrow G$, i.e. we have $\mathsf{Grp}(H, \ker(f)) \cong H_f$. Any universal property you have encountered can be expressed this way (or in covariant way).

Notice that we said "the" universal element. This suggests some kind of relations between universal elements, and so between classifying objects. We start by the following proposition, that will be crucial subsequently.

Proposition 1.7.5. Let $F, G : \mathcal{C}^{\mathsf{op}} \to \mathsf{Set}$ be functors represented by C_0 and C'_0 with natural isomorphisms $e : \mathcal{C}(-, C_0) \Rightarrow F$ and $e' : \mathcal{C}(-, C'_0) \Rightarrow G$. If there is a natural transformation $\kappa : F \Rightarrow G$, then there exists a unique morphism $\rho : C_0 \to C'_0$ such that the following diagram commutes:

$$\begin{tikzcd} \mathcal{C}(C, C_0) \arrow[r, "\rho_*"] \arrow[d, "e_C"', "\cong"] & \mathcal{C}(C, C'_0) \arrow[d, "e'_C", "\cong"'] \\ F(C) \arrow[r, "\kappa_C"] & G(C), \end{tikzcd} \tag{1.6}$$

for any object C in $\mathcal{C}$. Moreover, if κ is a natural isomorphism, then ρ is an isomorphism in $\mathcal{C}$.

Proof. Let us first define $\rho : C_0 \to C_0'$. The universal element of F is given by: $u_F = e_{C_0}(\mathrm{id}_{C_0}) \in F(C_0)$. Taking its image with κ_{C_0}, we obtain an element $\kappa_{C_0}(u_F)$ in $G(C_0)$. Since $e'_{C_0} : \mathcal{C}(C_0, C_0') \to G(C_0)$ is a bijection, there is a unique element ρ in $\mathcal{C}(C_0, C_0')$, such that $e'_{C_0}(\rho) = \kappa_{C_0}(u_F)$.
We now prove the commutativity of the diagram (1.6). Let φ be any morphism in $\mathcal{C}(C, C_0)$. On the one hand we have:

$$\begin{aligned} \kappa_C \circ e_c(\varphi) &= \kappa_C F(\varphi)(u_F), \text{ by Yoneda Lemma,} \\ &= G(\varphi)\kappa_{C_0}(u_F), \text{ by naturality of } \kappa, \\ &= G(\varphi)e'_{C_0}(\rho), \text{ by definition of } \rho, \end{aligned}$$

and on the other hand, we have:

$$\begin{aligned} e'_C \circ \rho_*(\varphi) &= e'_C(\rho \circ \varphi), \\ &= G(\rho \circ \varphi)(u_G), \text{ by Yoneda Lemma,} \\ &= G(\varphi) \circ G(\rho)(u_G) \\ &= G(\varphi)e'_{C_0}(\rho), \text{ by Yoneda Lemma.} \end{aligned}$$

We have just proved that the commutativity of diagram (1.6). Uniqueness of ρ follows immediately from its construction since ρ is the unique morphism making the diagram commute in the case $C = C_0$.

Let κ be a natural isomorphism. This means that for any object C in $\mathcal{C}$, the morphism $\kappa_C : F(C) \to G(C)$ is bijective. So there exists an inverse $\kappa_C^{-1} : G(C) \to F(C)$, for each object C. This obviously defines a natural transformation $\overline{\kappa} : G \Rightarrow F$, where $\overline{\kappa_C} = \kappa_C^{-1}$. Applying the first part of this proof, there is a unique morphism $\overline{\rho} : C_0' \to C_0$ corresponding to $\overline{\kappa}$. Moreover, we have $\overline{\kappa_C} \circ \kappa_C = \mathrm{id}_{F(C)}$ and $\kappa_C \circ \overline{\kappa_C} = \mathrm{id}_{G(C)}$, for every object C. But these composites of natural transformations correspond respectively to $\overline{\rho} \circ \rho$ and $\rho \circ \overline{\rho}$. By uniqueness, we obtain $\overline{\rho} \circ \rho = \mathrm{id}_{C_0}$ and $\rho \circ \overline{\rho} = \mathrm{id}_{C_0'}$, and so ρ is an isomorphism. □

We can now prove that classifying objects are unique up to isomorphism.

Corollary 1.7.7. Let $F : \mathcal{C}^{\mathsf{op}} \to \mathsf{Set}$ be a representable functor. If C_0 and C_0' are representing objects of F with universal elements u_F and u_F' respectively, then there is an isomorphism $\rho : C_0 \to C_0'$ in $\mathcal{C}$ such that $F(\rho)(u_F') = u_F$.

Proof. There are natural isomorphisms $e : \mathcal{C}(-, C_0) \Rightarrow F$ and $e' : \mathcal{C}(-, C_0') \Rightarrow F$. Taking the composites, we obtain another natural transformation: $\lambda := e'^{-1} \circ e : \mathcal{C}(-, C_0) \Rightarrow \mathcal{C}(-, C_0')$, which is obviously a natural isomorphism. By Proposition 1.7.5, λ determines a unique isomorphism $\rho : C_0 \to C_0'$, such that $\lambda_C(f) = \rho \circ f$, for any object C and morphism $f : C \to C_0$. In particular $\lambda_{C_0}(\mathrm{id}_{C_0}) = \rho$.
The universal elements u_F and u_F' are given respectively by $e_{C_0}(\mathrm{id}_{C_0})$ and $e'_{C_0'}(\mathrm{id}_{C_0'})$. We get:

$$\begin{aligned} F(\rho)(u_F') &= F(\rho) \circ e'_{C_0}(\mathrm{id}_{C_0'}) \\ &= e'_{C_0}(\rho), \text{ by naturality of } e', \\ &= e'_{C_0} \circ \lambda_{C_0}(\mathrm{id}_{C_0}) \\ &= e_{C_0}(\mathrm{id}_{C_0}), \text{ since } e' \circ \lambda = e, \\ &= u_F, \end{aligned}$$

and so $F(\rho)(u_F') = u_F$ as desired. □

Corollary 1.7.8. The Yoneda lemma determines a fully faithful embedding:

$$\begin{aligned} \mathcal{Y} \colon \mathcal{C} &\longrightarrow \mathrm{Fun}(\mathcal{C}^{\mathsf{op}}, \mathsf{Set}) \\ C_0 &\longmapsto \mathcal{C}(-, C_0) \\ \left(C_0 \xrightarrow{f} C_0'\right) &\longmapsto \left(\mathcal{C}(-, C_0) \overset{f_*}{\Rightarrow} \mathcal{C}(-, C_0')\right). \end{aligned}$$

Here f_* is the natural transformation defined for all $C \in \mathcal{C}$ as the set map $f_* \colon \mathcal{C}(C, C_0) \to \mathcal{C}(C, C_0')$ where $f_*(\varphi) = f \circ \varphi$ for all $\varphi \colon C \to C_0$ in $\mathcal{C}$.

What this corollary is saying is that in particular, we have for any two objects X and Y in $\mathcal{C}$ that:

$$X \cong Y \Leftrightarrow \mathcal{C}(C, X) \cong \mathcal{C}(C, Y), \forall C \in \mathcal{C}.$$

Chapter 2

Homotopy theories

2.1 Simplicial sets

Let Δ denote the simplex category, whose objects are ordered sets of the form $[n] = \{0, 1, \ldots, n\}$, and whose morphisms are order-preserving maps. The morphisms of Δ are generated by *cofaces* and *codegeneracies*. For example, cofaces are of the form

$$d^0, d^1 : [0] \to [1],$$

skipping 0 or 1 in $[1]$, etc. The codegeneracies look like $s^0 : [1] \to [0]$ which "repeat" an element. The cofaces and codegeneracies satisfy certain *cosimplicial identities*.

If $\mathcal{C}$ is a category, we let $s\mathcal{C} = \mathcal{C}^{\Delta^{\mathrm{op}}}$ denote the category of simplicial objects in $\mathcal{C}$. If $\mathcal{C} = \mathsf{Set}$, we write $\mathrm{sSet} := \mathsf{Set}^{\Delta^{\mathrm{op}}}$ and call it the category of simplicial sets. A simplicial set $X_\bullet \in \mathrm{sSet}$ consists of sets $X_0, X_1, \ldots$ together with face and degeneracy maps satisfying the simplicial identities.

Example 2.1.1 (The *nerve of a small category*)**.** Let $\mathcal{C} \in \mathsf{Cat}$ be a small category. We let $N_\bullet\mathcal{C}$ denote the simplicial set with $N_0\mathcal{C} = \mathrm{Ob}\mathcal{C}$, $N_1\mathcal{C} = \mathrm{Mor}\mathcal{C}$, and $N_n\mathcal{C}$ the set of n composable morphisms in $\mathcal{C}$. That is,

$$N_n\mathcal{C} = N_1\mathcal{C} \times_{N_0\mathcal{C}} \cdots \times_{N_0\mathcal{C}} N_1\mathcal{C}.$$

The face maps are given by source/target/composition, and the degeneracies insert an identity morphism.

Example 2.1.2. By the Yoneda embedding, we get a functor

$$\Delta^n := \mathrm{Hom}_\Delta(-, [n]) : \Delta^{\mathrm{op}} \to \mathsf{Set}.$$

If $X_\bullet$ is a simplicial set, then the set of n-simplices X_n is in bijection with $\mathrm{Hom}_{\mathrm{sSet}}(\Delta^n, X_\bullet)$.

Example 2.1.3 (Dold–Kan)**.** There is an isomorphism $\mathrm{Ch}_R^{\geq 0} \xrightarrow{\Gamma} s\mathrm{Mod}_R$, where $\Gamma_m C_\bullet = \oplus_{[n]\twoheadrightarrow[k]} C_k$. The faces and degeneracies are left as an exercise.

Example 2.1.4. Define $\Delta^n_{\mathrm{Top}} \subseteq \mathbb{R}^{n+1}$ by

$$\left\{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} : 0 \leq t_i \leq 1,\ \sum t_i = 1\right\}.$$

View $[n] = \{v_0, \ldots, v_n\}$, for $v_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ with a 1 at the ith place. Then if $\alpha\colon [m] \to [n]$ in Δ, we can define $\alpha(v_i) = v_{\alpha(i)}$. Extend linearly to get $\alpha_*\colon \Delta^m_{\mathrm{Top}} \to \Delta^n_{\mathrm{Top}}$. Then $\Delta^\bullet_{\mathrm{Top}}$ is a cosimplicial topological space.

Example 2.1.5. If $X \in \mathrm{Top}$, we can define a simplicial set $\mathrm{Sing}_\bullet(X) \in \mathrm{sSet}$ by $\mathrm{Sing}_n(X) = \mathrm{Hom}_{\mathrm{Top}}\left(\Delta^n_{\mathrm{Top}}, X\right)$.

Definition 2.1.6. If $X_\bullet \in \mathrm{sSet}$, its *geometric realization* is the topological space

$$|X_\bullet| = \amalg_{n\geq 0} X_n \times \Delta^n_{\mathrm{Top}}/\sim,$$

where $(x, s) \sim (y, t)$ if and only if there is some $\alpha\colon [m] \to [n]$ so that $\alpha^* y = x$ and $\alpha_* s = t$.

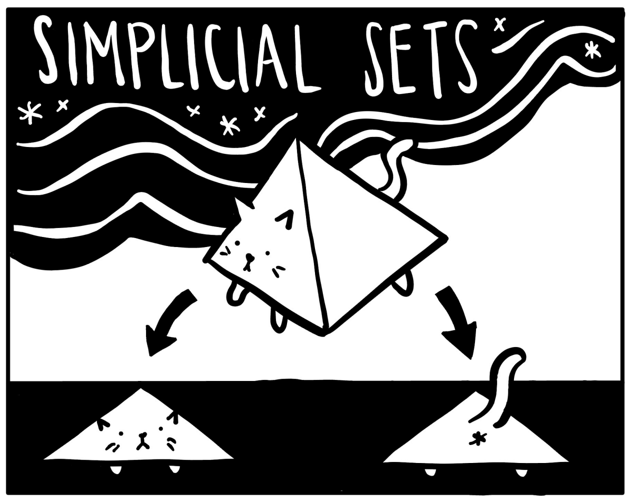
Example 2.1.7. For $n \geq 0$, $|\Delta^n_\bullet| \cong \Delta^n_{\mathrm{Top}}$.

Exercise 2.1.8. For any simplicial set X, $|X_\bullet|$ is always a CW complex.

Exercise 2.1.9. There is an adjunction $|-| : \mathrm{sSet} \rightleftarrows \mathrm{Top} : \mathrm{Sing}(-)$

Definition 2.1.10. A map $X_\bullet \to Y_\bullet$ is a *weak homotopy equivalence* in sSet if $|X_\bullet| \xrightarrow{\sim} |Y_\bullet|$ is a weak homotopy equivalence of spaces.

Theorem 2.1.11 (Quillen)**.** Simplicial sets up to weak equivalence is equivalent to topological spaces up to weak homotopy equivalence. Moreover, for any $X \in \mathrm{Top}$, $|\mathrm{Sing}(X)|$ is weakly equivalent to X.



The homotopy hypothesis (continued). If we are interested in studying Top up to weak homotopy equivalences, we may equivalently study sSet up to weak equivalence; the relationship between the two categories was given by the geometric realization / singular complex adjunction.

Recall that $\Delta^n = \mathrm{Hom}_\Delta(-, [n])$. We define the *kth horn* $\Lambda^n_k \subseteq \Delta^n$ as a coequalizer in sSet

$$\left(\coprod_{0\leq i<j\leq n} \Delta^{n-2} \rightrightarrows \coprod_{i\neq k} \Delta^{n-1} \right) \to \Lambda^n_k,$$

where the two maps are δ^{j-1} and δ^i. The geometric realization of Λ^n_k is the topological n-simplex, with the middle and the face opposite the kth edge removed.

Definition 2.1.12. A simplicial set $Y \in \mathrm{sSet}$ is a *Kan complex* if for all $k \leq n$, and for every $\Lambda^n_k \to Y$, there exists a (not necessarily unique) lift:

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & Y \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

Exercise 2.1.13. A simplicial set Y is a Kan complex if and only if for any $(n-1)$-simplices $y_1, \ldots, y_{k-1}, y_{k+1}, \ldots, y_n$ such that $d_i y_j = d_{j-1} y_i$ for $i < j$, $i, j \neq k$, there exists an n-simplex y such that $d_i y = y_i$ for all $i \neq k$.

Exercise 2.1.14. The simplicial set $\mathrm{Sing}(X)$ is always a Kan complex for any $X \in \mathrm{Top}$.

Exercise 2.1.15. The simplicial set Δ^n is not a Kan complex for $n \geq 1$.

Exercise 2.1.16. If $X \in s\mathsf{Grp}$, then the underlying simplicial set of X is always a Kan complex.

We will see later that, up to weak homotopy equivalence, every simplicial set is a Kan complex.
Recall the Dold-Kan correspondence

$$s\mathrm{Mod}_{\mathbb{Z}} \cong \mathrm{Ch}_{\mathbb{Z}}^{\geq 0},$$

which sends weak homotopy equivalences to quasi-isomorphisms. Given a simplicial set X_*, we can take an associated simplicial abelian group $\mathbb{Z}[X_*]$ by taking the free group on n-simplices at level n. We can ask what $\mathbb{Z}[X_*]$ corresponds to as a chain complex. One answer is that

$$\mathbb{Z}[\mathrm{Sing}(X_*)] \leftrightarrow C_*(X;\mathbb{Z}),$$

which tells us that

$$\pi_*\left(\mathbb{Z}\left[\mathrm{Sing}(X)\right]\right) \cong H_*(X;\mathbb{Z}).$$

In some sense, we can view $\mathbb{Z}[\mathrm{Sing}(X)]$ as being (equivalent to) the *free commutative monoid* on X. This statement is what is known as the *Dold-Thom theorem.*

Homotopy hypothesis: Spaces (up to weak equivalence) are ∞-groupoids. For us, spaces up to weak equivalences correspond to Kan complexes.

Given $X \in \mathrm{Kan}$, we can call X_0 the objects, and X_1 the morphisms. The horn filling conditions imply that we can *compose* and *invert* morphisms in X_1, witnessed by simplices in X_2.

Definition 2.1.17. A *quasi-category* (i.e. ∞-category) is a simplicial set with inner horn lifting property. That is, we can lift against horns Λ_k^n for $0 < k < n$.

Exercise 2.1.18. A quasi-category has unique horn filling if and only if it is isomorphic to the nerve of a 1-category.

2.2 Model structures

Vista: Every nice infinity category is equivalent (in some sense) to a model category.

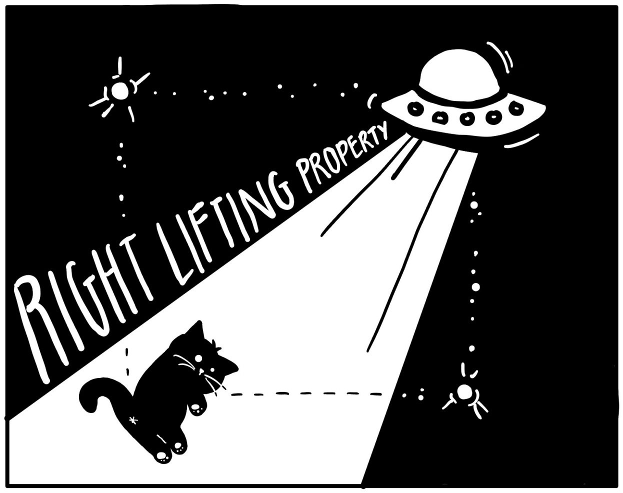
Notation 2.2.1. Let $\mathcal{M}$ be a category, and $\chi \subseteq \mathcal{M}$ a class of morphisms. We define $\mathrm{LLP}(\chi)$ to be the class of morphisms in $\mathcal{M}$ so that f has the *left lifting property* with respect to all morphisms in χ:

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d, "f"'] & \cdot \arrow[d, "\in \chi"] \\ \cdot \arrow[r] \arrow[ur, dashed] & \cdot \end{tikzcd}$$

Similarly we can define $f \in \mathrm{RLP}(\chi)$ by

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d, "\chi \ni"'] & \cdot \arrow[d, "f"] \\ \cdot \arrow[r] \arrow[ur, dashed] & \cdot \end{tikzcd}$$

Definition 2.2.2. A *weak factorization system* on a category $\mathcal{M}$ consists of a pair $(\mathcal{C}, \mathcal{F})$ of classes of morphisms such that



1. Given any $f : X \to Y$ in $\mathcal{M}$, it factors (not necessarily uniquely) as

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\mathcal{C}\ni"'] & & Y \\ & W \arrow[ur, "\in\mathcal{F}"'] & \end{tikzcd}$$

2. $\mathcal{C} = \mathrm{LLP}(\mathcal{F})$ and $\mathcal{F} = \mathrm{RLP}(\mathcal{C})$.

Example 2.2.3. In Set, the monomorphisms and epimorphisms give a weak factorization system. A factorization is

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\mathrm{id}_X\times f"'] & & Y \\ & X\times Y \arrow[ur, "\pi_Y"'] & \end{tikzcd}$$

Definition 2.2.4. A *model structure* on $\mathcal{M}$ consists of three classes of morphisms:

W	weak equivalences
Cof	cofibrations
Fib	fibrations

We use the notation $\widetilde{\mathrm{Cof}} := \mathrm{Cof} \cap W$ and $\widetilde{\mathrm{Fib}} = \mathrm{Fib} \cap W$, and call these *trivial cofibrations* (resp. *trivial fibrations*). These collections of morphisms are subject to the constraint that

1. $\mathcal{M}$ is bicomplete (all limits and colimits)[1]

2. W satisfies 2-out-of-3 property[2]

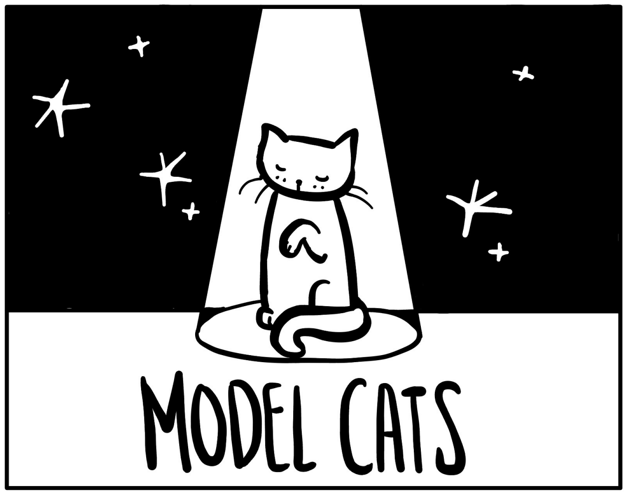
3. $\left(\mathrm{Cof}, \widetilde{\mathrm{Fib}}\right)$ and $\left(\widetilde{\mathrm{Cof}}, \mathrm{Fib}\right)$ are weak factorization systems.

Terminology 2.2.5. A category with a model structure is referred to as a *model category.*

Notation 2.2.6. We will decorate each class of morphisms as

[1]We might also require *finitely* bicomplete.

[2]If f and g are composable, and any two of f, g, gf are in W then so is the third.



W	$\xrightarrow{\sim}$
Cof	$\hookrightarrow$
Fib	$\twoheadrightarrow$

Exercise 2.2.7. The collections W, Cof, and Fib are closed under retracts: that is, if we have a diagram

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d, "f"'] \arrow[rr, bend left, equal] & \cdot \arrow[r] \arrow[d, "g"] & \cdot \arrow[d, "f"] \\
\cdot \arrow[r] \arrow[rr, bend right, equal] & \cdot \arrow[r] & \cdot
\end{tikzcd}$$

then if $g \in W$ (resp. Cof or Fib) then $f \in W$ (resp. Cof or Fib).

Definition 2.2.8. Let $\mathcal{M}$ be a model category, and let $\emptyset \in \mathcal{M}$ the initial object and $* \in \mathcal{M}$ the terminal object.

- We say that $X \in \mathcal{M}$ is *cofibrant* if the unique map $\emptyset \to X$ is a cofibration.
- We say that $X \in \mathcal{M}$ is *fibrant* if the unique map $X \to *$ is a fibration.
- We say that $\widetilde{X}$ is a *cofibrant replacement* of X if

$$\begin{tikzcd}
\emptyset \arrow[rr] \arrow[dr, hook] & & X \\
& \widetilde{X} \arrow[ur, two heads, "\sim"'] &
\end{tikzcd}$$

- We say that $\widetilde{X}$ is a *fibrant replacement* of X if

$$\begin{tikzcd}
X \arrow[rr] \arrow[dr, hook, "\sim"'] & & * \\
& \widetilde{X} \arrow[ur, two heads] &
\end{tikzcd}$$

Example 2.2.9. Let $\mathcal{M} = \text{Top}$, W = weak homotopy equivalences, Cof = relative CW complexes[3] The fibrations are determined by Fib = RLP($\widetilde{\text{Cof}}$) or, equivalently, RLP($D^n \to D^n \times I$). Every object is fibrant, and the cofibrant objects are precisely the CW complexes. Cofibrant replacement is cellular approximation.

[3] $A \hookrightarrow X$ is a *relative CW complex* if X is built out of A by attaching cells.

Proposition 2.2.10. Identities and isomorphisms are weak equivalences in a model category.

Proof. For any $X \in \mathcal{M}$, we can fibrantly replace it to get $X \overset{\sim}{\hookrightarrow} \widetilde{X}$. Consider the commutative diagram

$$\begin{tikzcd} X \arrow[rr, "\mathrm{id}"] \arrow[dr, hook, "\sim"'] & & X \arrow[dl, "\sim"] \\ & \widetilde{X}. & \end{tikzcd}$$

By 2-out-of-3, the identity $\mathrm{id}\colon X \to X$ is also a weak equivalence. More generally if $f\colon X \to Y$ is an isomorphism in $\mathcal{M}$, then by the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend left, equal] \arrow[d, "f"'] & Y \arrow[r, "f^{-1}"] \arrow[d, equal] & X \arrow[d, "f"] \\ Y \arrow[r, equal] & Y \arrow[r, equal] & Y, \end{tikzcd}$$

we see that f is contained in W. ◻

If $(\mathcal{C}, \mathcal{F})$ is a weak factorization system, then both $\mathcal{C}$ and $\mathcal{F}$ are closed under retracts. Hence Cof, $\widetilde{\mathrm{Cof}}$, Fib, $\widetilde{\mathrm{Fib}}$ are closed under retracts. As an exercise, show that W is also closed under retracts.

Exercise 2.2.11. A category $\mathcal{M}$ is a model category if and only if $\mathcal{M}^{\mathrm{op}}$ is a model category.

Theorem 2.2.12. Cofibrations are closed under pushouts and coproducts.

Proof. Given any test square, we can try to lift:

$$\begin{tikzcd} X \arrow[r] \arrow[d, hook] & Y \arrow[r] \arrow[d] & A \arrow[d, two heads, "\sim"] \\ Z \arrow[r] \arrow[urr, dashed] & P \arrow[r] \arrow[ul, phantom, "\ulcorner"] & B. \end{tikzcd}$$

This map is constructed by universal property of the pushout:

$$\begin{tikzcd} X \arrow[r] \arrow[d, hook] & Y \arrow[d] \arrow[ddr, bend left] & \\ Z \arrow[r] \arrow[drr, dashed, bend right] & P \arrow[ul, phantom, "\ulcorner"] \arrow[dr, dashed, "\exists!"] & \\ & & A. \end{tikzcd}$$

For coproducts, we can take $X_i \hookrightarrow Y_i$ for $i \in J$. Let's try to lift:

$$\begin{tikzcd} X_i \arrow[r] \arrow[d, hook] & \amalg_i X_i \arrow[r] \arrow[d] & A \arrow[d, two heads, "\sim"] \\ Y_i \arrow[r] \arrow[urr, dashed] & \amalg_i Y_i \arrow[r] & B. \end{tikzcd}$$

We know that each $X_i \hookrightarrow Y_i$ is a cofibration hence it lifts against the big square. By universal property a map $\amalg_i Y_i \to A$ exists. ◻

Example 2.2.13. If $\mathcal{C}$ is a bicomplete category, then $\mathcal{C}$ has a model structure where W is the isomorphisms, and Cof = Fib = Mor$\mathcal{C}$.

Example 2.2.14. If $\mathcal{M}$ = Top, the *Quillen model structure* is given by

- W = weak homotopy equivalences,
- Cof = retracts of relative CW complexes,
- Fib = Serre fibrations (RLP($D^n \hookrightarrow D^n \times I$)).

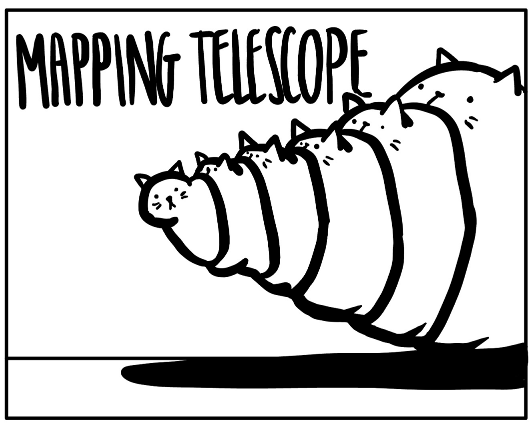
Example 2.2.15. The Strøm (or Hurewicz) model structure on Top is given by

- W = homotopy equivalences,
- Fib = Hurewicz fibrations (RLP($A \to A \times I$) for all $A \in$ Top),
- Cof = closed cofibrations in Top.

Fibrant replacement in the Strøm model structure looks like

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, hook] & & Y \\ & M_f \arrow[ur, two heads, "\simeq"'] & \end{tikzcd}$$

Where $M_f = (X \times I) \cup_X Y$ is the mapping cylinder.



Example 2.2.16. The *Kan model structure* on sSet is given by

- W = weak homotopy equivalences,
- Cof = monomorphisms (levelwise injections),
- Fib = Kan fibrations (RLP($\Lambda^n_k \to \Delta^n$) for all $0 \leq k \leq n$).

Everything is cofibrant here (since the empty simplicial set injects into everything). Fibrant objects are Kan complexes. Thus, every simplicial set is weakly equivalent to a Kan complex!

Theorem 2.2.17 (Milnor)**.** The natural map $X \to \mathrm{Sing}(|X|)$ is a weak homotopy equivalence for any simplicial set X. (See also Kerodon, 3.5.4.1.)

Definition 2.2.18. Let $\mathcal{C}$ be a category, and $W \subseteq \mathcal{C}$ a subcategory. A functor $F : \mathcal{C} \to \mathcal{D}$ is called the *localization of* $\mathcal{C}$ *with respect to* W if:

1. $F(f) \in \mathrm{iso}\mathcal{D}$ if $f \in \mathrm{Mor}W$,

2. For any other F' satisfying (1), we have

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F'"] \arrow[d, "F"'] & \mathcal{D}' \\ \mathcal{C} \arrow[ur, "\exists!"'] & \end{tikzcd} .$$

We let $\mathcal{C} \to \mathcal{C}[W^{-1}]$ denote the localization.

Here is a naive way to construct $\mathcal{C}[W^{-1}]$: we take the free category on $\mathcal{C}$ and "W^{-1}." That is, we take the same objects, but allow morphisms to be "zigzags" of morphisms forward in $\mathcal{C}$ and morphisms backwards in W, and we mod out by the relation that things in W become isomorphisms. There are size issues here.

Theorem 2.2.19. If $\mathcal{M}$ is a model category, then localization $\mathcal{M} \to \mathcal{M}[W^{-1}]$ exists. We denote by $\mathrm{Ho}(\mathcal{M}) = \mathcal{M}[W^{-1}]$ the homotopy category of $\mathcal{M}$.

Recall in Top that $f \simeq g : X \to Y$ if there is a map $H : X \times I \to Y$ so that $H(-,0) = f$ and $H(-,1) = g$.

Definition 2.2.20. Let $\mathcal{M}$ be a model category. A *cylinder object* on $X \in \mathcal{M}$ is defined to be

$$\begin{tikzcd} X \amalg X \arrow[rr, "\nabla"] \arrow[dr, hook] & & Y \\ & \mathrm{Cyl}(X) \arrow[ur, "\sim"'] & \end{tikzcd}$$

The construction of cylinder objects is *not functorial*.

A *(left) homotopy* from f to g is a map $H : \mathrm{Cyl}(X) \to Y$ such that $H \circ i_0 = f$ and $H \circ i_1 = g$. We denote this by $f \simeq g$.

Proposition 2.2.21. We have that $i_0 : X \to \mathrm{Cyl}(X)$ is a weak equivalence (and same for i_1).

Proof. We have

$$\begin{tikzcd} X \arrow[r] \arrow[rr, bend left, "\mathrm{id}"] \arrow[dr, dashed, "i_0"'] & X \amalg X \arrow[r, "\nabla"] \arrow[d] & Y \\ & \mathrm{Cyl}(X) \arrow[ur, "\sim"'] & \end{tikzcd}$$

By 2-out-of-3 on the outside maps, the result follows. □

Proposition 2.2.22. If X is cofibrant, then $i_0, i_1 : X \to \mathrm{Cyl}(X)$ are cofibrations.

Proof. Since cofibrations are preserved under pushouts, we have that i_0 and i_1 are cofibrations:

$$\begin{tikzcd} \emptyset \arrow[r, hook] \arrow[d, hook] & X \arrow[d, "i_0"] \\ X \arrow[r, "i_1"'] \arrow[ur, phantom, "\ulcorner"] & X \amalg X \end{tikzcd}$$

□

Theorem 2.2.23. (Exercise) If X is cofibrant, then homotopy $\simeq$ gives an equivalence relation on $\mathrm{Hom}(X, Y)$ for any Y.

We can think of a map

$$\begin{aligned}\mathrm{Hom}_{\mathcal{M}}(X,Y)/{\simeq} \times \mathrm{Hom}_{\mathcal{M}}(Y,Z)/{\simeq} &\to \mathrm{Hom}_{\mathcal{M}}(X,Z)/{\simeq} \\ (f,g) &\mapsto g \circ f.\end{aligned}$$

In order for this to be well-defined, we need Z to be fibrant.

Lemma 2.2.24. If Z is fibrant, and $f \simeq g : X \to Z$, then if $h : X' \to X$, we have that $fh \simeq gh$.

Proof. We have $H : \mathrm{Cyl}(X) \to Y$ with $H_0 = f$ and $H_1 = g$. By lifting, we get

$$\begin{tikzcd} X' \amalg X' \arrow[r] \arrow[d, hook] & X \amalg X \arrow[r] & \mathrm{Cyl}(X) \arrow[d, two heads, "\sim"] \\ \mathrm{Cyl}(X') \arrow[r] \arrow[urr, dashed] & X' \arrow[r] & X. \end{tikzcd}$$

This gives the desired map. We used fibrancy of Z to ensure that the map $\mathrm{Cyl}(X) \to X$ was a trivial fibration (or could be replaced with a better cylinder object using a map to Z). ◻

Theorem 2.2.25. In $\mathcal{M}$, given $f : X \to Y$ with X cofibrant and Y fibrant, then $f \in W$ if and only if f is a homotopy equivalence.[4]

Notation 2.2.26. $\mathcal{M}_c$ = cofibrant objects in $\mathcal{M}$, and $\mathcal{M}_f$ = fibrant objects in $\mathcal{M}$. We denote by $\mathcal{M}_{cf}$ = objects which are *both* cofibrant and fibrant.

Concretely, we can define $\mathrm{Ho}(\mathcal{M})$ as the objects in $\mathcal{M}$, but where

$$\mathrm{Hom}_{\mathrm{Ho}(\mathcal{M})}(X,Y) = \mathrm{Hom}_{\mathcal{M}_{cf}/\simeq}(RQX, RQY),$$

where R is a fibrant replacement and Q is a cofibrant replacement.

Exercise 2.2.27. Given $X \to Y$ in $\mathcal{M}$, there exists $QX \xrightarrow{\widetilde{f}} QY$ such that

$$\begin{tikzcd} QX \arrow[r, "\widetilde{f}"] \arrow[d, two heads, "\sim"] & QY \arrow[d, two heads, "\sim"] \\ X \arrow[r, "f"'] & Y. \end{tikzcd}$$

Here $\widetilde{f}$ is well-defined up to left homotopy.

Given some $\mathcal{M} \to \mathrm{Ho}(\mathcal{M})$, we just need to check that $W \mapsto$ isos, and it is universal in that way.

2.3 Derived functors

Definition 2.3.1. Suppose $\mathcal{M}$ and $\mathcal{N}$ are model categories, and take a functor $F : \mathcal{M} \to \mathcal{N}$. A *left derived functor* of F is an (absolute) right Kan extension of F along $\gamma_{\mathcal{M}} : \mathcal{M} \to \mathrm{Ho}(\mathcal{M})$:

$$\begin{tikzcd} \mathcal{M} \arrow[r, "F"] \arrow[d, "\gamma_{\mathcal{M}}"'] & \mathcal{N} \\ \mathrm{Ho}(\mathcal{M}) \arrow[ur, dashed] \arrow[u, phantom, shift left=4, "\Leftarrow \ell"] \end{tikzcd}$$

[4] Meaning that there is some $g : Y \to X$ with $fg \simeq \mathrm{id}$ and $gf \simeq \mathrm{id}$.

if $G : \mathrm{Ho}(\mathcal{M}) \to \mathcal{N}$ and $s : G \circ \gamma_{\mathcal{M}} \Rightarrow F$, then there exists a unique $s' : G \Rightarrow LF$ so that $\ell \circ (s' \circ \gamma_{\mathcal{M}}) = s$.

$$\begin{tikzcd} \mathcal{M} \arrow[r, "F"] \arrow[d, "\gamma_{\mathcal{M}}"'] & \mathcal{N} \\ \mathrm{Ho}(\mathcal{M}) \arrow[ur, dashed] \arrow[ur, bend right] \arrow[ur, phantom, "\Leftarrow \ell" near end, shift left=2ex] \arrow[ur, phantom, "\Uparrow s'", shift right=1ex] & \end{tikzcd}$$

Definition 2.3.2. Let $F : \mathcal{M} \to \mathcal{N}$. A *total left derived functor* $\mathbb{L}F : \mathrm{Ho}(\mathcal{M}) \to \mathrm{Ho}(\mathcal{N})$ is the left derived functor of $\mathcal{M} \xrightarrow{F} \mathcal{N} \xrightarrow{\gamma_{\mathcal{N}}} \mathrm{Ho}(\mathcal{N})$.

Example 2.3.3. If $\mathcal{F} : \mathcal{M} \to \mathcal{N}$ where if $f \in W$ between cofibrant objects then Ff is a weak equivalence in $\mathcal{N}$, then $\mathbb{L}F$ exists:

$$\begin{tikzcd} \mathcal{M} \arrow[r, "F"] \arrow[d] & \mathcal{N} \arrow[r] & \mathrm{Ho}(\mathcal{N}) \\ \mathrm{Ho}(\mathcal{M}) \arrow[urr, dashed, bend right] & & \end{tikzcd}$$

We will have that $\mathbb{L}F(X) \xrightarrow{\sim} F(X)$ whenever X is cofibrant. In general, $\mathbb{L}F(X) = F(Q(X))$.

Definition 2.3.4. Let $F : \mathcal{M} \to \mathcal{N}$. We say that F is a *left Quillen functor* if

(i) F is a left adjoint

(ii) F preserves cofibrations and trivial cofibrations.

In this case if G is a right adjoint, then we say the adjunction is a *Quillen adjunction / Quillen pair*.[5]

Exercise 2.3.5. Show that L is left Quillen if and only if G is right Quillen.

Lemma 2.3.6. (Ken Brown's Lemma) If $F : \mathcal{M} \to \mathcal{N}$ is any functor between model categories which sends trivial cofibrations between cofibrant objects to weak equivalences in $\mathcal{N}$, then F sends any weak equivalence between cofibrant objects to weak equivalences.

Proof. Let $f : A \xrightarrow{\sim} B$, where $A, B \in \mathcal{M}_c$. We need $F(f)$ to be a weak equivalence. Consider the factorization of the coproduct of f and the identity on B:

$$\begin{tikzcd} A \amalg B \arrow[rr, "f \amalg \mathrm{id}_B"] \arrow[dr, hook, "q"'] & & B \\ & C \arrow[ur, two heads, "p"', "\sim"] & \end{tikzcd}$$

Then consider the pushout:

$$\begin{tikzcd} \emptyset \arrow[r, hook] \arrow[d, hook] & A \arrow[r, "f"] \arrow[d, hook, "i_A"] \arrow[ddr, "q"', "\sim"] & B \\ B \arrow[r, hook] \arrow[ddrr, equal] \arrow[drr, "q"] & A \amalg B \arrow[dr, "q"] & \\ & & C \arrow[uu, "p"'] \arrow[d, "p"] \\ & & B \end{tikzcd}$$

---

[5] There is a dual notion of right Quillen functor, meaning it is a right adjoint which preserves fibrations and trivial fibrations.

We have that

$$B \overset{i_B}{\hookrightarrow} A \amalg B \overset{q}{\hookrightarrow} C$$
$$A \overset{i_A}{\hookrightarrow} A \amalg B \overset{q}{\hookrightarrow} C$$

are both trivial cofibrations, hence their images under F are weak equivalences. We see that

$$F(p) \circ F(q \circ \mathrm{id}_B) = F(p \circ q \circ \mathrm{id}_B) = F(\mathrm{id}_B).$$

Therefore $F(p)$ is a weak equivalence by 2-out-of-3. $\square$

Theorem 2.3.7. Suppose that $F : \mathcal{M} \to \mathcal{M}$ is left Quillen. Then $\mathbb{L}F : \mathrm{Ho}(\mathcal{M}) \to \mathrm{Ho}(\mathcal{N})$ exists and can be defined as

$$\mathrm{Ho}(\mathcal{M}) \xrightarrow{Q} \mathrm{Ho}(\mathcal{M}_c) \xrightarrow{F} \mathrm{Ho}(\mathcal{N}).$$

Moreover, we obtain an adjunction on the homotopy categories:

$$\mathbb{L}F : \mathrm{Ho}(\mathcal{M}) \rightleftarrows \mathrm{Ho}(\mathcal{N}) : \mathbb{R}G.$$

Proof idea. We have a natural iso

$$\mathrm{Hom}_{\mathcal{M}}(X, G(Y)) \cong \mathrm{Hom}_{\mathcal{N}}(F(X), Y),$$

compatible with homotopy equivalence:

$$\mathrm{Hom}_{\mathcal{M}}(X, G(Y))/{\simeq} \cong \mathrm{Hom}_{\mathcal{N}}(F(X), Y)/{\simeq}$$

$\square$

Theorem/Definition: Take a Quillen adjunction $F : \mathcal{M} \rightleftarrows \mathcal{N} : G$. Suppose that $f : X \xrightarrow{\sim} G(Y)$, with $X \in \mathcal{M}_c$ and $Y \in \mathcal{N}_f$ is a weak equivalence if and only if $f^\flat : F(X) \to Y$ is. Then $\mathbb{L}F$ and $\mathbb{R}G$ are equivalences of categories, we call this a *Quillen equivalence.*

Example 2.3.8. We have that

$$|-| : \mathrm{sSet}_{\mathrm{Kan}} \rightleftarrows \mathrm{Top}_{\mathrm{Quillen}} : \mathrm{Sing}(-)$$

is a Quillen equivalence.

Example 2.3.9. We have that

$$\mathrm{id} : \mathrm{Top}_{\mathrm{Quillen}} \rightleftarrows \mathrm{Top}_{\mathrm{Strøm}} : \mathrm{id}$$

is a Quillen adjunction but not a Quillen equivalence.

Q: If $\mathcal{M}$ and $\mathcal{N}$ are model categories such that there is an equivalence of categories $\mathrm{Ho}(\mathcal{M}) \cong \mathrm{Ho}(\mathcal{N})$, is this always coming from a Quillen equivalence?

A: No! Dugger–Shipley, 2009.

This indicates that Quillen equivalence is a good notion but it is not a *perfect* notion.

2.4 Guided example: chain complexes

Let's take $\mathrm{Ch}_{\mathbb{Z}}$ to be homologically graded unbounded chain complexes. There are three model structures of interest. We first start with the projective one:

$(\mathrm{Ch}_{\mathbb{Z}})_{\mathrm{projective}}$:

- weak equivalences are quasi-isomorphisms
- fibrations are levelwise epimorphisms
- cofibrations are levelwise monomorphisms such that the cokernel of each $f_n : X_n \to Y_n$ is free.

If $M \in \mathsf{Ab}$, we define $S^n(M)$ to be the chain complex $M[n]$ which is concentrated in M at degree n. If $M = \mathbb{Z}$, we call it S^n. We define $D^n(M)$ to be a chain complex

$$\cdots \to 0 \to M \xrightarrow{\mathrm{id}} M \to 0 \to \cdots$$

with two M's concentrated in degrees n and $n-1$. We call $D^n(\mathbb{Z}) =: D^n$.

Exercise 2.4.1. Show that fibrations are $\mathrm{RLP}(0 \to D^n)$ for all n. That is,

$$\begin{tikzcd} 0 \arrow[r] \arrow[d, hook] & X \arrow[d, two heads] \\ D^n \arrow[r] \arrow[ur, dashed] & Y. \end{tikzcd}$$

We claim this lifts iff $X \to Y$ is a levelwise epimorphism. We have that $\mathrm{Hom}_{\mathrm{Ch}}(D^n, Y) \cong Y_n$, so we are just asking if every element in Y_n lifts to an element in X_n.

Exercise 2.4.2. Show that $\widetilde{\mathrm{Fib}} = \mathrm{RLP}\left(S^n \hookrightarrow D^{n+1}\right)$ for all n. Consider $\mathrm{Hom}_{\mathrm{Ch}}(S^n, Y)$. A map looks like

$$\begin{tikzcd} \cdots \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & Y_n \arrow[r] & Y_{n-1} \arrow[r] & \cdots \end{tikzcd}$$

That is, it picks out a class in Y_n which maps to zero under the differential. The data of a square

$$\begin{tikzcd} S^{n-1} \arrow[r] \arrow[d, hook] & X \arrow[d, "p"] \\ D^n \arrow[r] & Y \end{tikzcd}$$

is the data of $(y, x) \in Y_n \oplus Z_{n-1}X$ so that $p(x) = dy$. Show that a lift exists if and only if p is a trivial fibration.

Other model structures.
$(\mathrm{Ch}_R)_{\text{injective}}$:

- W = quasi-isomorphisms
- Cof = fiberwise monomorphisms[6]
- Fib = fiberwise epimorphisms with fibrant kernel

We get a Quillen equivalence

$$\mathrm{id} : (\mathrm{Ch}_R)_{\text{projective}} \rightleftarrows (\mathrm{Ch}_R)_{\text{injective}} : \mathrm{id}.$$

We also have have a third one which is *not* Quillen equivalent.
$(\mathrm{Ch}_R)_{\text{Hurewicz}}$:

- W = homotopy equivalences of chain complexes
- Cof = split levelwise monomorphisms
- Fib = split levelwise epimorphisms

We denote by $\mathcal{D}(R) = \mathrm{Ho}\left((\mathrm{Ch}_R)_{\text{proj}}\right)$ the *derived category* of a ring R.

We can also think about *connective* chain complexes (which are zero in negative degrees). We have an adjunction

$$\mathrm{Ch}_R \rightleftarrows \mathrm{Ch}_R^{\geq 0}.$$

This induces a model structure on $\mathrm{Ch}_R^{\geq 0}$ making it into a Quillen adjunction but not a Quillen equivalence. We denote by $\mathrm{Ho}(\mathrm{Ch}_R^{\geq 0}) = \mathcal{D}^{\geq 0}(R)$.

We get a model structure: $\left(\mathrm{Ch}_R^{\geq 0}\right)_{\text{proj}}$

- W = quasi-isomorphisms
- Fib = positive epimorphisms (may not be epi in degree 0)
- Cof = monomorphisms with projective cokernel. The cofibrant objects here are levelwise projective R-modules.

If we take $M \in \mathrm{Mod}_R$, we can view $S^0(M) \in \mathrm{Ch}_R^{\geq 0}$, and take a cofibrant replacement of it $P \xrightarrow{\sim} S^0(M)$. This is *exactly* a projective resolution of M!

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r] \arrow[d] & P_1 \arrow[r] \arrow[d] & P_0 \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & M \arrow[r] & 0.
\end{tikzcd}$$

Example 2.4.3. Let $M \in \mathrm{Mod}_R$. Then we can take

$$S^0(M) \otimes_R - : \mathrm{Ch}_R^{\geq 0} \to \mathrm{Ch}_R^{\geq 0}.$$

We can check that this is left Quillen. We can look at its total left derived functor $S^0(M) \otimes_R^{\mathbb{L}} -$. We can see that

$$M \otimes_R^{\mathbb{L}} N := S^0(M) \otimes_R^{\mathbb{L}} S^0(N) \simeq S^0(M) \otimes_R P_\bullet,$$

where $P_\bullet$ is a projective resolution of N. We have that

$$H_i(M \otimes_R^{\mathbb{L}} N) = \mathrm{Tor}_i^R(M, N).$$

[6] Here we roughly have that $\mathrm{Cof} = \mathrm{LLP}(D^n \to 0)$ and $\widetilde{\mathrm{Fib}} = \mathrm{LLP}(D^{n+1} \to S^n)$.

Exercise 2.4.4. In the same way, if we want to derive hom, we can check that

$$\mathrm{Hom}_{\mathcal{D}^{\geq 0}(R)}(S^m(M), S^n(N)) \cong \mathrm{Ext}_R^{n-m}(M, N).$$

Via Dold-Kan, we have a Quillen adjunction

$$R[-] : \mathrm{sSet}_{\mathrm{Kan}} \rightleftarrows \mathrm{sMod}_R : U,$$

with the model structure on sMod_R given by weak homotopy equivalences as underlying simplicial sets, and fibrations as underlying Kan fibrations.

Then Dold-Kan takes the form of a Quillen equivalence

$$N : (\mathrm{sMod}_R)_{\mathrm{Kan}} \rightleftarrows (\mathrm{Ch}_R^{\geq 0})_{\mathrm{proj}} : \Gamma.$$

In general $N(X \otimes_R Y) \not\cong N(X) \otimes_R N(Y)$, however $N(X \otimes Y) \cong N(X) \otimes_R N(Y)$. They both describe $\mathcal{D}^{\geq 0}(R)$ in a monoidal way.

For Dold–Kan $\mathrm{Ch}_{\geq 0} \cong \mathrm{sMod}_R$, we have

$$M \otimes N \rightleftarrows M \otimes R \otimes N \rightleftarrows M \otimes R^{\otimes 2} N \cdots$$

we denote this by $B_\bullet(M, R, N)$ and call it the *bar construction.*

2.5 Homotopy (co)limits

Motivation: Limits and colimits are not invariant under (weak) homotopy equivalence.

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, hook] & CX \arrow[d] \\ CX \arrow[r] & \Sigma X \arrow[ul, phantom, "\ulcorner"] \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & * \arrow[ul, phantom, "\ulcorner"] \end{tikzcd}$$

However $\Sigma X \not\simeq *$.

Let $\mathcal{M}$ be a model category, and $\mathcal{C}$ a small category. Then we denote by $\mathrm{Fun}(\mathcal{C}, \mathcal{M}) = \mathcal{M}^{\mathcal{C}}$. Let $\mathcal{C}_0 \subsetneq \mathcal{C}$ be the discrete subcategory spanned by $\mathrm{Ob}(\mathcal{C})$. Let $\mathcal{M}^{\mathcal{C}_0} = \prod_{\mathcal{C}_0} \mathcal{M}$. This has a model structure where W, Fib, and Cof are determined objectwise.

Consider $\iota : \mathcal{C}_0 \hookrightarrow \mathcal{C}$. This induces a map

$$\begin{aligned} \iota^* : \mathcal{M}^{\mathcal{C}} &\to \mathcal{M}^{\mathcal{C}_0} \\ F &\mapsto F|_{\mathcal{C}_0} . \end{aligned}$$

This admits adjoints:

$$\iota_! \dashv i^* \dashv i_*.$$

We have that ι^* creates W and Fib.

We have $\left(\mathcal{M}^{\mathcal{C}}\right)_{\mathrm{proj}}$:

- W = objectwise weak equivalence
- Fib = objectwise fib
- Cof = ? induced by $\iota_!$Cof

We have that $\mathcal{M}$ is cocomplete, so we get a tensoring

$$\begin{aligned} \mathcal{M} \times \mathsf{Set}^{\mathcal{C}} &\to \mathcal{M}^{\mathcal{C}} \\ (X, F) &\mapsto X \otimes F = \amalg_{F(-)} X. \end{aligned}$$

We have $(X \times F)(c) = \amalg_{F(c)} X$.

There are representable functors

$$\begin{aligned} \mathcal{C}(c, -) : \mathcal{C} &\to \mathsf{Set} \\ d &\mapsto \mathcal{C}(c, d). \end{aligned}$$

By Yoneda, there is a natural iso

$$\mathsf{Set}^{\mathcal{C}}(\mathcal{C}(c, -), F) \cong F(c).$$

Tensoring with a representable functor gives

$$X \otimes \mathcal{C}(c, -) = \amalg_{\mathcal{C}(c,-)} X.$$

This is the *free diagram of X generated at c*.

This gives an adjunction

$$- \otimes \mathcal{C}(c, -) : \mathcal{M} \rightleftarrows \mathcal{M}^{\mathcal{C}} : \mathrm{ev}_c\,.$$

In this case

$$\iota_!(F) = \amalg_c \amalg_{\mathcal{C}(c,-)} F(c),$$

which is the free diagram in $\mathcal{M}$ generated by F. Evaluating at d gives

$$\iota_!(F)(d) = \amalg_{c \in \mathcal{C}} \amalg_{\mathcal{C}(c,d)} F(c).$$

This is the functor $\iota_! : \mathcal{M}^{\mathcal{C}_0} \to \mathcal{M}^{\mathcal{C}}$. We see that $\iota_! X$ is a left Kan extension

$$\begin{tikzcd} \mathcal{C}_0 \arrow[r, "X"] \arrow[d, hook, "\iota"'] & \mathcal{M} \\ \mathcal{C} \arrow[ur, dashed] & \end{tikzcd}$$

There is a diagonal functor

$$\begin{aligned} \mathcal{M} &\xrightarrow{\Delta} \mathcal{M}^{\mathcal{C}} \\ C &\mapsto \text{constant functor at } X. \end{aligned}$$

This admits adjoints

$$\operatorname{colim} \dashv \Delta \dashv \lim\,.$$

Proposition 2.5.1. The adjunction

$$\operatorname{colim} : \left(\mathcal{M}^{\mathcal{C}}\right)_{\text{proj}} \rightleftarrows \mathcal{M} : \Delta$$

is Quillen.

We denote hocolim := $\mathbb{L}$colim. There is a map hocolim$(-) \to$ colim$(-)$, and

$$\operatorname{hocolim}(F) \simeq \operatorname{colim}(QF).$$

Here QF denotes a cofibrant replacement in $\left(\mathcal{M}^{\mathcal{C}}\right)_{\text{proj}}$. For a general $\mathcal{C}$, QF is very difficult to determine.

Consider $\mathcal{C} = a \leftarrow b \to c$, and let $X \in \mathcal{M}^{\mathcal{C}_0}$. Then $\iota_! X$ is equal to

$$\begin{tikzcd} X(b) \arrow[r] \arrow[d] & X(b) \amalg X(c) \\ X(a) \amalg X(b) & \end{tikzcd}$$

Cofibrant objects in $\mathcal{M}^{\mathcal{C}}$ are of the form

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, hook] & Z \\ Y & \end{tikzcd}$$

with X cofibrant. Here cofibrant replacement is easy. We start with $Y \xleftarrow{f} X \xrightarrow{g} Z$, and we replace X with $\widetilde{X} \xrightarrow{\sim} X$ to get

$$\begin{tikzcd} \widetilde{X} \arrow[r] \arrow[d] & Y \\ Z & \end{tikzcd}$$

If we cofibrantly replace $\widetilde{X} \to Z$, and similarly for Y, we get

$$\begin{tikzcd} \widetilde{X} \arrow[r] \arrow[d] & \widetilde{Z} \\ \widetilde{Y} & \end{tikzcd}$$

The maps we used to fibrantly replace induces a fiberwise weak equivalence between this diagram and the one we started out with.

In $(\mathrm{Top})_{\text{Quillen}}$, we can take hocolim$(* \leftarrow X \to *)$. We cofibrantly replace X if necessary, and replace $X \to *$ by $X \hookrightarrow CX$, which is a cofibration. In this case we see that

$$\operatorname{hocolim}(* \leftarrow X \to *) \simeq \operatorname{colim}(C\widetilde{X} \leftarrow \widetilde{X} \to C\widetilde{X}) = \Sigma \widetilde{X}.$$

More generally, hocolim$(Y \xleftarrow{f} X \xrightarrow{g} Z)$ is the double mapping cylinder $M(f,g)$.

Theorem 2.5.2. If $\mathcal{M}$ is a *left proper model category* then

$$\operatorname{hocolim}(Y \hookleftarrow X \to Z) \cong \operatorname{colim}(Y \hookleftarrow X \to Z).$$

Proof. In the easy case, X is cofibrant, so we can factor the map to Z to get

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, hook] & \widetilde{Z} \arrow[r, two heads, "\sim"] \arrow[d] & Z \arrow[d] \\ Y \arrow[r] & H \arrow[r, dashed] \arrow[ur, phantom, "\ulcorner"] & P. \end{tikzcd}$$

The entire rectangle is a pushout, so $Z \to P$ is a cofibration, and the right square is a pushout by the pasting law, so $H \to P$ is a weak equivalence. ◻

Example 2.5.3. Let $\mathcal{C} = * \to * \to \cdots$. Show that $X_0 \to X_1 \to \cdots$ is cofibrant in $\mathcal{M}^{\mathcal{C}}$ if and only if X_0 is cofibrant and $X_i \hookrightarrow X_{i+1}$ is a cofibration for each i.

There is a third model structure on $\mathcal{M}^{\mathcal{C}}$ called the *Reedy model structure* (need $\mathcal{C}$ to be a Reedy cat). In this case, $\operatorname{hocolim}_{\Delta^{\mathrm{op}}}(X_\bullet) \cong \left|Q^{\mathrm{Reedy}} X_\bullet\right|$, for X a simplicial object in $\mathcal{M}$.

Bar construction: Let $\mathcal{M}$ a model cat, $\mathcal{C}$ a small cat, $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{M}$, and $G : \mathcal{C} \to \mathcal{M}$. Then we define

$$B_\bullet(F, \mathcal{C}, G) := \amalg_{c_0 \in \mathcal{C}} F(c_0) \times G(c_0) \leftleftarrows \amalg_{c_0 \leftarrow c_1} F(c_0) \times G(c_1) \leftleftarrows \cdots$$

Example 2.5.4. If $F = * = G$, then

$$B_\bullet(*, \mathcal{C}, *) \cong N_\bullet(\mathcal{C}^{\mathrm{op}}).$$

Pièce de résistance:

Theorem 2.5.5. (Bousfield–Kan) If $F : \mathcal{C} \to \mathcal{M}$ is a functor, then

$$\operatorname{hocolim}_{\mathcal{C}}(F) \simeq |B_\bullet(*, \mathcal{C}, F)|.$$

2.6 Combinatorial model categories

Show how model categories are enriched in spaces up to homotopy types and have cellular approximations

Definition 2.6.1. A model category is *combinatorial* if it is *presentable*[7] and *cofibrantly generated.*

To motivate presentability, let X be a set. Then X is determined by its elements, meaning that

$$\operatorname{Hom}_{\mathsf{Set}}(*, X) \cong X.$$

Then we can present X as $X = \cup_{x \in X} \{*\}$.

Definition 2.6.2. A colimit is *filtered* if the diagram is filtered, meaning it is nonempty and every subdiagram has a cocone.

Theorem 2.6.3. (Exercise) In Set, filtered colimits commute with finite limits. That is, if $F : I \times J \to \mathsf{Set}$ with I finite and J filtered, then

$$\operatorname{colim}_J \left(\lim_I F_I \right) \xrightarrow{\sim} \lim_I \left(\operatorname{colim}_J F_J \right)$$

is an isomorphism.

Proposition 2.6.4. A set X is finite if and only if

$$\operatorname{Hom}_{\mathsf{Set}}(X, -) : \mathsf{Set} \to \mathsf{Set}$$

preserves filtered colimits.

Proof. For the backwards direction, let $I = \{X_i\}$ be the collection of finite subsets of X. Then $X = \operatorname{colim}_I X_i$. In particular, we have that

$$\operatorname{colim}_I \operatorname{Hom}(X, X_i) \cong \operatorname{Hom}(X, X)$$
$$(X \xrightarrow{f_i} X_i) \xrightarrow{\sim} \mathrm{id}_X?$$

For the forwards direction, $\operatorname{Hom}_{\mathsf{Set}}(*, -) \cong \mathrm{id}_{\mathsf{Set}}$ so it preserves colimits. Since X is finite, we have that $X = \{x_1, \ldots, x_n\}$, hence

$$\operatorname{Hom}(X, -) \cong \operatorname{Hom}(\cup_i \{x_i\}, -) \cong \lim_i \operatorname{Hom}(\{x_i\}, -).$$

Then we use finite limits commuting with filtered colimits. ◻

[7] By this we mean "locally presentable."

Definition 2.6.5. An object $X \in \mathcal{C}$ is *compact* if $\mathrm{Hom}_{\mathcal{C}}(X, -) : \mathcal{C} \to \mathsf{Set}$ preserves filtered colimits.

Hence if $F : I \to \mathcal{C}$, with I filtered, then a map $X \to \mathrm{colim}_I F$ factors through an $F(i)$.

Examples 2.6.6. Compact objects:

- Set, compact = finite set
- Vect_F, compact = finite dimensional
- Mod_R, compact = finitely presented
- Grp, compact = finitely presented
- Top, compact = finite sets with discrete topology
- Ch, compact = perfect chain complexes (bounded, levelwise finitely generated and projective)
- sSet, compact = finite simplicial sets (X_n finite for each n, and there exists an m so that all non-degenerate simplices have dimension $\leq m$).

A topological space is (topologically) compact if and only if $X \in \mathcal{O}(X)$ is (categorically) compact.

Lemma 2.6.7. Finite colimits of compact objects are compact.

Definition 2.6.8. A category $\mathcal{C}$ is *presentable* if

1. $\mathcal{C}$ is cocomplete
2. There exists a set S of compact objects in $\mathcal{C}$ such that every object in $\mathcal{C}$ is a filtered colimit of objects in S.

We also say the "ind-completion" of S is $\mathcal{C}$, denoted $\mathrm{Ind}(S) = \mathcal{C}$.

Theorem 2.6.9. $\mathcal{C}$ is presentable if and only if there is an adjunction of the form

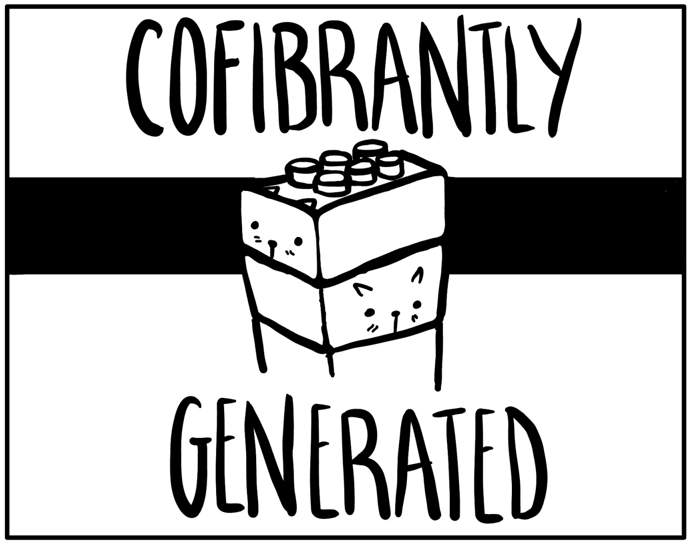
$$\mathrm{Fun}(K^{\mathsf{op}}, \mathsf{Set}) \rightleftarrows \mathcal{C},$$

where K is some small category, and the right adjoint is fully faithful and preserves filtered colimits.

We might take K for example to to be isomorphism classes of compact objects in $\mathcal{C}$, then we have

$$\begin{aligned} \mathcal{C} &\to \mathrm{Fun}(K^{\mathsf{op}}, \mathsf{Set}) \\ X &\mapsto \left(K^{\mathsf{op}} \to \mathcal{C}\mathsf{op} \xrightarrow{\mathrm{Hom}(-,X)} \mathsf{Set} \right). \end{aligned}$$

Theorem 2.6.10. Suppose $\mathcal{C}$ and $\mathcal{D}$ presentable. Then $L : \mathcal{C} \to \mathcal{D}$ preserves colimits if and only if L is a left adjoint.



Definition 2.6.11. Let I be a set of maps in a cocomplete category, fix λ to be an ordinal, and let $X : \lambda \to \mathcal{C}$ a functor, and suppose that $X(\alpha) \to X(\alpha+1)$ fits into

$$\begin{tikzcd} A_\alpha \arrow[r] \arrow[d] & X(\alpha) \arrow[d] \\ B_\alpha \arrow[r] & X(\alpha+1), \end{tikzcd}$$

where $A_\alpha \to B_\alpha$ is in I. Then we say that $X(0) \to \operatorname{colim}_\lambda X$ is a *relative I-cell complex.* We say an object $Y \in \mathcal{C}$ is an *I-cell complex* if $\emptyset \to Y$ is a relative I-cell complex.

If $I = \{S^n \hookrightarrow D^{n+1}\}_{n\geq 0}$, then we are recovering the idea of CW complexes in spaces.

We denote by $\mathrm{Cell}_I(\mathcal{C})$ the class of relative I-cell complexes.

Exercise 2.6.12. We have that $\mathrm{Cell}_I(\mathcal{C})$ is the smallest class in $\mathcal{C}$ closed under composition, pushouts, and filtered colimits.

Theorem 2.6.13. *(Small object argument)* Let $\mathcal{C}$ be cocomplete, let I a set of maps in $\mathcal{C}$, and suppose that for all $A \to B$ in I, we have that A is compact with respect to the full subcategory of of I-cells in $\mathcal{C}$. Then there exists a functorial factorization of maps in $\mathcal{C}$:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\gamma"'] & & Y \\ & C \arrow[ur, "\delta"'] & \end{tikzcd}$$

with $\gamma \in \mathrm{Cell}_I(\mathcal{C})$ and $\delta \in \mathrm{RLP}(I)$.

Proof idea. Start with $X(0) = X$, and take a map $X(0) \to Y$. Suppose $X(\beta) = \operatorname{colim}_{\alpha<\beta} X(\alpha)$ is constructed with $X(\beta) \to Y$. Look at the set[8]

$$S = \left\{ \begin{tikzcd} A \arrow[r] \arrow[d, "g"'] & X(\beta) \arrow[d] \\ B \arrow[r] & Y \end{tikzcd} : g \in I \right\}.$$

[8]Note this set is nonempty because we can take g to be $\mathrm{id} : X(\beta) \to X(\beta)$.

Denote by g_s the map $A \to B$ appearing in $s \in S$. Then we build

$$\begin{tikzcd} \amalg_{s\in S} A_s \arrow[r] \arrow[d, "\amalg_s g_s"'] & X(\beta) \arrow[d, "\in \mathrm{Cell}_I(\mathcal{C})"] \\ \amalg_{s\in S} B_s \arrow[r] \arrow[ur, phantom, "\ulcorner" very near end] & X(\beta+1) \end{tikzcd}$$

By UP of the pushout, there is an induced map $X(\beta+1) \to Y$. Then we claim that

$$X(0) \to \operatorname{colim}_\beta X(\beta) =: C$$

is in $\mathrm{Cell}_I(\mathcal{C})$. The only thing left to show is that $C \to Y$ is in $\mathrm{RLP}(I)$. Take

$$\begin{tikzcd} A \arrow[r] \arrow[d] & C = \operatorname{colim}_\beta X(\beta) \arrow[d] \\ B \arrow[r] & Y. \end{tikzcd}$$

Since A is compact with respect to I-cells, the map $A \to C$ factors through some $X(\beta)$. Since $B \to Y$ factors through $X(\beta+1)$, we see that it lifts to $B \to C$. □

Definition 2.6.14. A model category $\mathcal{M}$ is *cofibrantly generated* if there exist sets of maps I, J in $\mathcal{M}$ so that

- Cof = retracts of I-cell complexes, denoted $\widehat{\mathrm{Cell}_I(\mathcal{C})}$[9]
- Cof = $\widehat{\mathrm{Cell}_J(\mathcal{C})}$

and "I and J permit the small object argument."

Example 2.6.15. For $\mathrm{Top}_{\mathrm{Quillen}}$, we can take

$$I = \left\{S^n \hookrightarrow D^{n+1}\right\}$$
$$J = \{D^n \to D^n \times [0,1]\}.$$

Example 2.6.16. For $\mathrm{sSet}_{\mathrm{Kan}}$, we can take

$$I = \{\partial\Delta^n \to \Delta^n\}$$
$$J = \left\{\Lambda^k_n \to \Delta^n\right\}.$$

Example 2.6.17. For $(\mathrm{Ch}_R)_{\mathrm{proj}}$,

$$I = \left\{S^n \to D^{n+1}\right\}$$
$$J = \{0 \to D^n\}.$$

Example 2.6.18. The Strøm model structure is not cofibrantly generated in the definition above.

Theorem 2.6.19. (Kan — Right transfer) Let $\mathcal{M}$ be a cofibrantly generated model category and $\mathcal{C}$ is any category where there is an adjunction

$$F : \mathcal{M} \rightleftarrows \mathcal{C} : G.$$

Then $\mathcal{C}$ has a model structure where W and Fib are created by G. The model structure is cofibrantly generated by $F(I)$ and $F(J)$ if:

1. $F(I)$ and $F(J)$ permit the small object argument

[9] The hat $\widehat{\ }$ means "retracts of -"

2. $G\left(\mathrm{Cell}_{F(J)}\right)$ are weak equivalences in $\mathcal{M}$.

For combinatorial model categories, we get an inductive argument for building cofibrant replacements.

[Rezk-Schwede-Shipley] Combinatorial model categories are always simplicially enriched.

[Dugger] Any combinatorial model category $\mathcal{M}$ is Quillen equivalent to a localization of a projective Kan one:

$$L_\tau \mathrm{Fun}(K^{\mathsf{op}}, \mathrm{sSet}) \rightleftarrows \mathcal{M}.$$

2.7 Multiplicative structures on homotopy theories

The goal is to give a monoidal structure on the homotopy category $\mathrm{Ho}(\mathcal{M})$ of a model category $\mathcal{M}$, then we can consider rings and modules up to homotopy.

Definition 2.7.1. We say a symmetric monoidal category $(\mathcal{C}, \otimes, \mathbb{I})$ is *presentably symmetric monoidal* if:

- the category $\mathcal{C}$ is presentable;
- the bifunctor $-\otimes -\colon \mathcal{C}\times\mathcal{C}\to\mathcal{C}$ preserves colimits in each variable.

One consequence of being a presentably symmetric monoidal, is that the induced functor $X\otimes -\colon \mathcal{C}\to\mathcal{C}$ has a right adjoint, often denoted $[X,-]\colon \mathcal{C}\to\mathcal{C}$, i.e. the monoidal structure is closed.

Definition 2.7.2. We say a category $\mathcal{M}$ is a *(symmetric) monoidal model category* if we have the following.

1. The category $\mathcal{M}$ is endowed with a model structure.
2. It is presentably symmetric monoidal $(\mathcal{M}, \otimes, \mathbb{I})$. [10]
3. It respects the *pushout-product axiom*, which says that the bifunctor $-\otimes -\colon \mathcal{M}\times\mathcal{M}\to\mathcal{M}$ is a Quillen bifunctor, i.e. given cofibrations $f\colon X\hookrightarrow Y$ and $f'\colon X'\hookrightarrow Y'$ in $\mathcal{M}$, the induced dashed map $f\square f'$ on the pushout in $\mathcal{M}$

$$\begin{tikzcd}
X\otimes X' \arrow[r, "\mathrm{id}\otimes f'"] \arrow[d, "f\otimes\mathrm{id}"'] & X\otimes Y' \arrow[d] \arrow[ddr, bend left, "f\otimes\mathrm{id}"] & \\
Y\otimes X' \arrow[r] \arrow[drr, bend right, "\mathrm{id}\otimes f'"'] & P \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, hook, dashed, "f\square f'"] & \\
& & Y\otimes Y'
\end{tikzcd}$$

   is a cofibration in $\mathcal{M}$. Moreover, $f\square f'$ is a trivial cofibration as soon as f or f' is.
4. The monoidal unit $\mathbb{I}$ is cofibrant.[11]

Examples 2.7.3. $(\mathrm{sSet}_{\mathrm{Kan}}, \times, *)$, $((\mathrm{Ch}_R)_{\mathrm{proj}}, \otimes_R, R)$ and $(\mathrm{sMod}_R, \otimes_R, R)$ are symmetric monoidal model categories.

Examples 2.7.4. $(\mathrm{Top}_{\mathrm{Quillen}}, \times, *)$ and $((\mathrm{Ch}_R)_{\mathrm{inj}}, \otimes_R, R)$ are not monoidal model categories.

Exercise 2.7.5. Check that it is enough to verify the pushout-product axiom on the generating cofibrations and trivial cofibrations, if $\mathcal{M}$ is a cofibrantly generated model category.

[10] We may relax this condition and just ask the monoidal category to be closed.

[11] We may relax this condition and just ask that for some (hence any) cofibrant replacement $Q\mathbb{I}\to\mathbb{I}$ we get that $Q\mathbb{I}\otimes X\to \mathbb{I}\otimes X\cong X$ is a weak equivalence for any cofibrant object X.

Observe that one of the consequence of being a monoidal model category is that, for any cofibrant object $X \in \mathcal{M}$, the induced functor $X \otimes -\colon \mathcal{M} \to \mathcal{M}$ is a left Quillen functor (and thus $[X, -]\colon \mathcal{M} \to \mathcal{M}$ is a right Quillen functor). Indeed, given a cofibration $f'\colon A \to B$, denote by $f\colon \emptyset \hookrightarrow X$ the cofibration and apply the pushout product to $f \square f'$ we obtain the diagram:

$$\begin{tikzcd}
\emptyset \otimes A \arrow[r, "\exists!"] \arrow[d, "\exists!"'] & \emptyset \otimes B \arrow[d] \arrow[ddr, bend left, "\exists!"] & \\
X \otimes A \arrow[r] \arrow[ur, phantom, "\ulcorner"] \arrow[drr, bend right, "\mathrm{id}\otimes f'"'] & X \otimes A \arrow[dr, hook, dashed, "\mathrm{id}\otimes f'"] & \\
 & & X \otimes B
\end{tikzcd}$$

Since we assume $- \otimes Z$ to be a left adjoint for any object Z, it preserves initial object, so $\emptyset \otimes A \cong \emptyset \cong \emptyset \otimes B$. Thus $X \otimes -$ preserves cofibrations and trivial cofibrations. Therefore we can left derive the bifunctor given by the monoidal product, and we denote it

$$- \otimes^{\mathbb{L}} -\colon \mathrm{Ho}(\mathcal{M}) \times \mathrm{Ho}(\mathcal{M}) \longrightarrow \mathrm{Ho}(\mathcal{M}).$$

One can in fact check that we obtain the following.

Theorem 2.7.6 (Hovey)**.** Let $(\mathcal{M}, \otimes, \mathbb{I})$ be a (symmetric) monoidal model category. Then the derived tensor product endows the homotopy category $(\mathrm{Ho}(\mathcal{M}), \otimes, \mathbb{I})$ with a (symmetric) closed monoidal structure. Moreover, the localization functor $\mathcal{M} \to \mathrm{Ho}(\mathcal{M})$ is lax (symmetric) monoidal, and is strong monoidal if we restrict on cofibrant objects.

Similarly, we can introduce a variation on Quillen functors so that they are compatible with the monoidal structures.

Definition 2.7.7. A *weak symmetric monoidal Quillen pair* is a Quillen adjunction:

$$\begin{tikzcd}
(\mathcal{M}, \otimes, \mathbb{I}) \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & (\mathcal{N}, \wedge, \mathbb{J}) \arrow[l, bend left, "R"]
\end{tikzcd}$$

between symmetric monoidal model categories, for which L is oplax symmetric monoidal (or equivalently, R is lax symmetric monoidal) such that:

1. for all cofibrant objects X and Y in $\mathcal{M}$, the natural oplax map $L(X \otimes Y) \to L(X) \wedge L(Y)$ is a weak equivalence;
2. the natural map $L(\mathbb{I}) \to \mathbb{J}$ is a weak equivalence.

These two conditions are immediately verified if L is strong symmetric monoidal rather than just oplax symmetric monoidal. If the Quillen adjunction is a Quillen equivalence, then we refer to it as a *weak symmetric monoidal Quillen equivalence.*

Theorem 2.7.8 (Schwede-Shipley)**.** Given a weak symmetric monoidal Quillen pair

$$\begin{tikzcd}
(\mathcal{M}, \otimes, \mathbb{I}) \arrow[r, bend left] \arrow[r, phantom, "\perp"] & (\mathcal{N}, \wedge, \mathbb{J}) \arrow[l, bend left]
\end{tikzcd} ,$$

we obtain a weak symmetric monoidal adjunction on the homotopy categories:

$$\begin{tikzcd}
(\mathrm{Ho}(\mathcal{M}), \otimes^{\mathbb{L}}, \mathbb{I}) \arrow[r, bend left] \arrow[r, phantom, "\perp"] & (\mathrm{Ho}(\mathcal{N}), \wedge^{\mathbb{L}}, \mathbb{J}) \arrow[l, bend left]
\end{tikzcd}$$

It is an equivalence of symmetric monoidal categories if the original adjunction is a weak symmetric monoidal Quillen equivalence.

Example 2.7.9. (Schwede-Shipley) Regard the equivalence of categories $\mathrm{Ch}_R^{\geq 0} \cong \mathrm{sMod}_R$ as a weak symmetric monoidal Quillen adjunction:

$$\begin{tikzcd} (\mathrm{Ch}_R^{\geq 0})_{\mathrm{proj}} \arrow[r, bend left, "\Gamma"] \arrow[r, phantom, "\perp"] & \mathrm{sMod}_R \arrow[l, bend left, "N"] \end{tikzcd}$$

where we give the normalization functor a lax symmetric monoidal structure $N(A) \otimes N(B) \to N(A \otimes B)$ via the Eilenberg-Zilber map. This map is not an isomorphism, and so the equivalence $\mathrm{Ch}_R^{\geq 0} \cong \mathrm{sMod}_R$ is not compatible with the monoidal structures. However, once derived, since the above is a weak symmetric monoidal Quillen equivalence, we obtain that $\mathrm{Ho}(\mathrm{Ch}_R^{\geq 0}) \cong \mathrm{Ho}(\mathrm{sMod}_R)$ is an equivalence of symmetric monoidal categories. Indeed, we can show the Eilenberg-Zilber map has a homotopy inverse $N(A \otimes B) \to N(A) \otimes N(B)$ given by the Alexander-Whitney map.

Given a monoidal model category $\mathcal{M}$, we can also lift model structures on modules and algebras in $\mathcal{M}$. For instance, we can use Kan's right transfer to defined a right-induced model structure on the category $\mathrm{Alg}(\mathcal{M})$ of algebra objects in $\mathcal{M}$ using the free-forgetful adjunction:

$$\begin{tikzcd} \mathcal{M} \arrow[r, bend left, "T"] \arrow[r, phantom, "\perp"] & \mathrm{Alg}(\mathcal{M}) \arrow[l, bend left, "U"] \end{tikzcd}$$

where $T(X) = \bigoplus_{n \geq 0} X^{\otimes n}$. For this, not only we need to assume cofibrantly generated, we also require the following axiom.

Definition 2.7.10. We say a combinatorial symmetric monoidal model category $\mathcal{M}$ respect the *monoid axiom* if we have the following. Given any object X in $\mathcal{M}$, denote by $S_X = \{X \otimes A \overset{\mathrm{id} \otimes f}{\to} X \otimes B \mid f \in J\}$ where J is the set of generating trivial cofibrations. Then any relative S_X-cell complex is a weak equivalence. If all objects are cofibrant, the axiom is automatically verified.

By Kan's right transfer, in order to obtain a model structure on $\mathrm{Alg}(\mathcal{M})$ we would need to check that maps in $U(\mathrm{Cell}_{T(J)})$ are weak equivalences in $\mathcal{M}$. Thus we need first to understand transfinite composition in $\mathrm{Alg}(\mathcal{M})$. As T preserves filtered colimits, then U preserves and reflects filtered colimits. So we need to understand certain pushouts in $\mathrm{Alg}(\mathcal{M})$. These will be pushouts along free maps: given $f \colon X \to Y$ in $\mathcal{M}$, consider the pushout in $\mathrm{Alg}(\mathcal{M})$

$$\begin{tikzcd} T(X) \arrow[r] \arrow[d, "T(f)"] & T(Y) \arrow[d] \\ A \arrow[r] \arrow[ur, phantom, "\ulcorner", near end] & P. \end{tikzcd}$$

We can give a very explicit construction of P: it is the telescope in $\mathcal{M}$

$$P_0 = A \longrightarrow P_1 \longrightarrow P_2 \longrightarrow \cdots$$

that we describe below. Informally, one can think of P as the formal product of elements in A and in Y subject to the relations between letters induced by $f \colon X \to Y$ and the multiplication in A, while P_n only considers at most n factors from elements in Y. Let us now give a more robust definition.

Let us denote by $\mathcal{P}(\{1, \dots, n\})$ the poset of power set of the set with n-elements. Define a functor $W_n \colon \mathcal{P}(\{1, \dots, n\}) \to \mathcal{M}$ as follows: on objects $S \subseteq \{1, \dots, n\}$, let

$$W_n(S) = A \otimes C_1 \otimes A \otimes C_2 \otimes \cdots \otimes C_n \otimes A$$

where

$$C_i = \begin{cases} X & \text{if } i \notin S \\ Y & \text{if } i \in S \end{cases}$$

The assignment on the maps is induced by the map $f : X \to Y$. The functor W_n defines an n-dimensional cube diagram in $\mathcal{M}$. For instance, at $n = 2$, it looks like:

$$\begin{tikzcd} A \otimes X \otimes A \otimes X \otimes A \arrow[r] \arrow[d] & A \otimes X \otimes A \otimes Y \otimes A \arrow[d] \\ A \otimes Y \otimes A \otimes X \otimes A \arrow[r] & A \otimes Y \otimes A \otimes Y \otimes A \end{tikzcd}$$

Denote by $\widetilde{W}_n$ the restriction of the functor W onto the full subcategory of $\mathcal{P}(\{1, \ldots, n\})$ for which we removed the terminal object. Again, for $n = 2$, it looks like:

$$\begin{tikzcd} A \otimes X \otimes A \otimes X \otimes A \arrow[r] \arrow[d] & A \otimes X \otimes A \otimes Y \otimes A \\ A \otimes Y \otimes A \otimes X \otimes A & \end{tikzcd}$$

Let $Q_n = \operatorname{colim} \widetilde{W}_n$ in $\mathcal{M}$, and define P_n inductively (recall $P_0 = A$) as the pushout in $\mathcal{M}$:

$$\begin{tikzcd} Q_n \arrow[r] \arrow[d] & (A \otimes Y)^{\otimes n} \otimes A \arrow[d] \\ P_{n-1} \arrow[r] & P_n \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

The top horizontal map is induced by the universal property of the colimit and the maps we have removed from W_n to obtain $\widetilde{W}_n$. The left vertical map $Q_n \to P_{n-1}$ is defined by repeatedly applying the map $X \to A$ whenever $C_i = X$ in $\widetilde{W}_n(S)$, i.e. $i \notin S$, and then if $A \otimes A$ appears in the copy, use the multiplication on A.

We are now left to check 3 things:

1. P is an algebra in $\mathcal{M}$
2. the induced map $A \to P$ is an algebra homomorphism
3. P is indeed the desired pushout in $\mathrm{Alg}(M)$.

For (1): the unit of A induces the unit of P:

$$\mathbb{I} \to A = P_0 \to P = \operatorname{colim}_{n \geq 0} P_n$$

The multiplication on P is defined from maps $P_n \otimes P_m \to P_{n+m}$ which can be defined from the pushout definition of P_n by simply concatening all the words together. It its then elementary to show that the multiplication is indeed associative and unital. This also automatically shows (2). For (3), suppose there was an algebra B fitting into the diagram in $\mathrm{Alg}(\mathcal{M})$:

$$\begin{tikzcd} T(X) \arrow[r] \arrow[d] & T(Y) \arrow[d] \\ A \arrow[r] & B. \end{tikzcd}$$

By adjunction, it also defines a diagram in $\mathcal{M}$:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ A \arrow[r] & B. \end{tikzcd}$$

Define the unique homomorphism $P \to B$ of algebras by applying the maps $Y \to B$ and $A \to B$ whenever appropriate, this uniquely defines it. We are now ready to show the following.

Theorem 2.7.11. (Schwede-Shipley) Suppose $\mathcal{M}$ is a combinatorial symmetric monoidal model category that respects the monoid axiom, where the generating cofibrations and trivial cofibrations are denoted by (I, J) respectively. Then there exists a right-induced combinatorial model structure on $\mathrm{Alg}(\mathcal{M})$, i.e., fibrations and weak equivalences are created in $\mathcal{M}$ via the forgetful functor in the adjunction

$$\begin{tikzcd} \mathcal{M} \arrow[r, bend left, "T"] \arrow[r, phantom, "\perp"] & \mathrm{Alg}(\mathcal{N}) \arrow[l, bend left, "U"] \end{tikzcd}$$

The generating cofibrations and trivial cofibrations are $(T(I), T(J))$.

Proof. From Kan's right transfer theorem, we need to check maps in $U(\mathrm{Cell}_{T(J)})$ are weak equivalences. So suppose in our construction of P above that $f\colon X \overset{\sim}{\hookrightarrow} Y$ was a trivial cofibration. We need to show $A \to P$ is a weak equivalence. It is enough to show $P_{n-1} \to P_n$ is a weak equivalence for all $n \geq 1$. For this notice that the map $Q_n \to (A \otimes Y)^{\otimes n} \otimes A$ is isomorphic to $\overline{Q}_n \otimes A^{\otimes n+1} \to Y^{\otimes n} \otimes A^{\otimes n+1}$ using symmetry, where $\overline{Q}_n$ is obtained as Q_n but where we deleted all instances of A appearing in the punctured cube $\widetilde{W}_n$. Then using the pushout-product axiom, we can check that the induced map $\overline{Q}_n \to Y^{\otimes n}$ is a trivial cofibration. Thus by the monoid axiom, we get that $P_{n-1} \to P_n$ is a weak equivalence. $\square$

Exercise 2.7.12. Show that if A is cofibrant as an algebra in $\mathcal{M}$, then A is also cofibrant as an underlying object in $\mathcal{M}$.

A similar result can be deduced for modules, and it is easier as colimits of modules are computed in the underlying category.

Exercise 2.7.13. Suppose $\mathcal{M}$ is a combinatorial symmetric monoidal model category that respects the monoid axiom, where the generating cofibrations and trivial cofibrations are denoted by (I, J) respectively. Let R be an algebra in $\mathcal{M}$. Show that the category of right R-modules $\mathrm{Mod}_R(\mathcal{M})$ is combinotorial model category, with weak equivalences and fibrations determined in $\mathcal{M}$, and the generating cofibrations and trivial cofibrations are given by $I \otimes R$ and $J \otimes R$ respectively, using the free-forgetful adjunction

$$\begin{tikzcd} \mathcal{M} \arrow[r, bend left, "-\otimes R"] \arrow[r, phantom, "\perp"] & \mathrm{Mod}_R(\mathcal{M}) \arrow[l, bend left, "U"] \end{tikzcd}$$

Exercise 2.7.14. Show that if one additionally requires R to be a commutative algebra, then the induced model structures in $\mathrm{Mod}_R(\mathcal{M})$ in previous exercise is in fact a symmetric monoidal model structure that also satisfies the monoid axiom, with respect to the relative tensor product over R.

Exercise 2.7.15. Let $f\colon R \to S$ be a homomorphism of algebras in combinatorial symmetric monoidal model category $\mathcal{M}$ that respects the monoid axiom. Show there is a Quillen adjunction

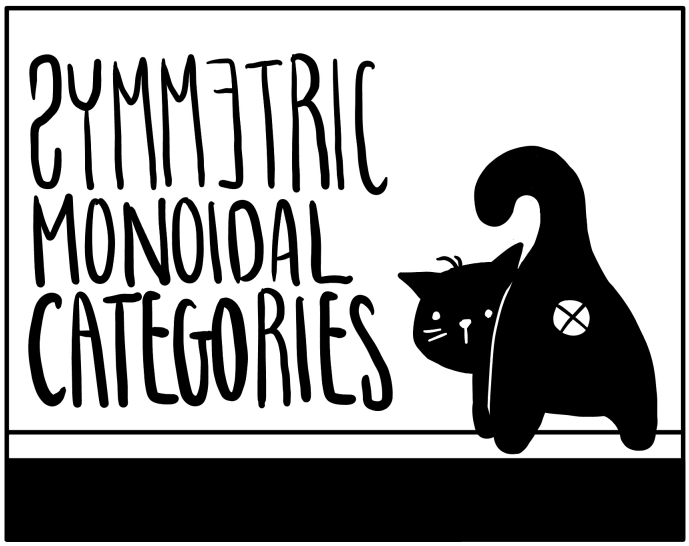
$$\begin{tikzcd} \mathrm{Mod}_R(\mathcal{M}) \arrow[r, bend left, "-\otimes S"] \arrow[r, phantom, "\perp"] & \mathrm{Mod}_S(\mathcal{M}) \arrow[l, bend left, "f^*"] \end{tikzcd}$$

Show it is a Quillen equivalence, if and only if f is a weak equivalence. Show it is (strong) monoidal Quillen pair if R and S are commutative.

Remark 2.7.16. The case for commutative algebra is more subtle. It is sometimes possible to lift the model structure as in the non-commutative case, but further restrictions on $\mathcal{M}$ is required. Notably, one can see that it is impossible to give a model structure right-induced on chains:

$$\begin{tikzcd} (\mathrm{Ch}_R)_{\mathrm{proj}} \arrow[rr, bend left] \arrow[rr, phantom, "\perp"] & & \mathrm{CAlg}(\mathrm{Ch}_R) \arrow[ll, bend left, "U"] \end{tikzcd}$$

whenever $\mathrm{char}(R) \neq 0$. Indeed, suppose $\mathrm{char}(R) = p$, and $A \to B$ is a homomorphism of commutative algebra, that is a fibration in Ch_R. Suppose $x \in H_n(B)$, for n even. Then x^p is in the image of $H_*(A) \to H_*(B)$. There exists $y \in A$ that is mapped to x but $dy^p = py^{p-1} = 0$ by Leibniz rule. Therefore it is impossible to factor a homomorphism of commutative algebra by a weak equivalence followed by a fibration.



2.8 Application: homotopy coherent multiplication on spaces

Last time: We had $\mathcal{M}$ a model category, and $\otimes$ a monoidal structure. We used this to give a monoidal structure on $\mathrm{Ho}(\mathcal{M})$, given by $\otimes^{\mathbb{L}}$, the *left derived tensor product*. We used this to give a homotopy theory on $\mathrm{Alg}(\mathcal{M})$, and $\mathrm{Mod}_R(\mathcal{M})$, etc.

Q: What are algebras in the homotopy category of a model structure $\mathcal{M}$? An example of interest is $\mathcal{M} = \mathrm{Top}$.

What are commutative algebras in Top?

Theorem 2.8.1. (Moore) If $X \in \mathrm{CAlg}(\mathrm{Top})$, then there is a weak equivalence

$$\prod_{i=1}^{\infty} K(\pi_i(X), i) \to X.$$

Proof. Let $G_n = \pi_n(X)$. Then we take

$$0 \to F \to \mathbb{Z}[G_n] \to G_n \to 0.$$

Then we get that $\widetilde{H}_n(\vee_{g \in G_n} S^n) \cong \oplus_{g \in G_n} \widetilde{H}_n(S^n) = \mathbb{Z}[G_n]$. Using the Hurewicz theorem, there is an isomorphism

$$\pi_n(\vee S^n) \xrightarrow{\sim} \widetilde{H}_n(\vee S^n),$$

so we can pick $f_j \in \pi_n(S^n)$ for each e_j in a basis of F. This gives us a pushout

$$\begin{tikzcd} \vee_{j\in J} S^n \arrow[r] \arrow[d] & \vee_{g\in G_n} S^n \arrow[d] \\ * \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & M(G_n, n) \end{tikzcd}$$

This gives a map $\vee_{n\geq 1} M(G_n, n) \to X$. By universal property, we get an algebra homomorphism[12][13]

$$\mathrm{SP}(\vee_{n\geq 1} M(G_n, n)) \to X$$

The Dold–Thom theorem states that $\pi_* \mathrm{SP}(Y) \cong \widetilde{H}_*(Y)$, given some connectedness hypothesis (path-connected?). We get that

$$\mathrm{SP}(\vee_{n\geq 1} M(G_n, n)) \cong \prod_n \mathrm{SP}(M(G_n, n)) = \prod_n K(G_n, n).$$

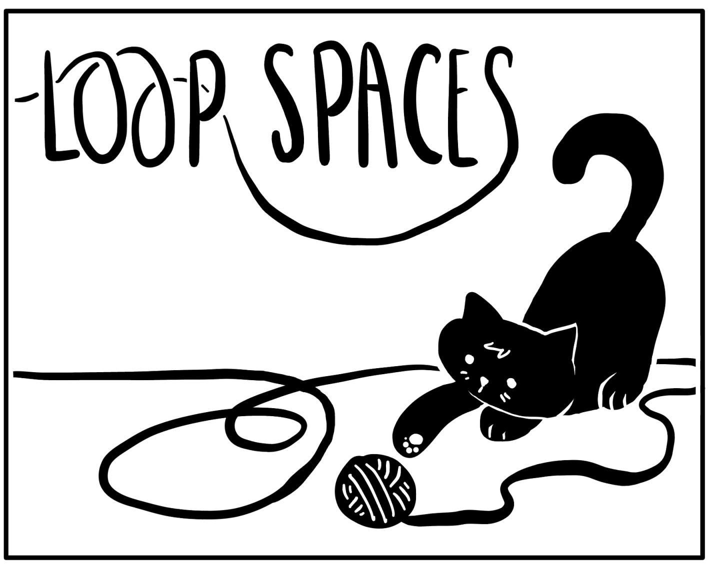
$\square$

Definition 2.8.2. We say that $X \in \mathrm{Alg}(\mathrm{Ho}(\mathrm{Top}))$ if and only if X is a CW complex, with multiplication and unit

$$X \times X \to X$$
$$* \to X$$

which are associative and unital *up to homotopy*.

These are also called *H-spaces*. The most prototypical example is a loop space.



Example 2.8.3. If X is a based space, we can build ΩX as the homotopy pullback of the two maps from a point. Concatenation gives a map $\Omega X \times \Omega X \to \Omega X$.

Example 2.8.4. Eilenberg-MacLane spaces $K(G, n)$ are uniquely determined up to homotopy. We have that

$$\pi_k(\Omega K(G, n)) \cong \pi_{k+1}(K(G, n))$$

therefore $\Omega K(G, n) = K(G, n-1)$.

[12] Here $\mathrm{SP}(-)$ denotes the infinite symmetric product, i.e. the free commutative algebra in Top.
[13] The infinite symmetric product is left adjoint to the forgetful functor, i.e. $\mathrm{SP} : \mathrm{Top} \rightleftarrows \mathrm{CAlg}(\mathrm{Top}) : U$.

Q: Given X an H-space, such that $\pi_0 X$ is a group, is X a loop space?

A: No, there are many grouplike H-spaces that are not equivalent to ΩX. For example $S^7 \subseteq \mathbb{O}$ the unit octonians.

Loop spaces have an extra condition. Given $w, x, y, z \in \Omega X$, there is an association $(xy)z \simeq x(yz)$. There is a pentagon witnessing the different ways to associate four elements.

We can keep going with 5 loops, 6 loops... and we get the Stasheff associahedra $K(n)$, which tell us how to concatenate n loops. These give maps

$$K(n) \times (\Omega X)^n \to \Omega X,$$

witnessing the higher associativities of concatenation. We call this an A_∞-algebra structure.

Theorem 2.8.5. (Stasheff) Given X connected, we have that $X \simeq \Omega Y$ for some Y if and only if X is an A_∞-algebra in spaces that is grouplike.

Rigidification: We have that $\mathrm{Ho}(\mathrm{Alg}(\mathrm{sSet}, \times)) \simeq \mathrm{Alg}_{A_\infty}(\mathrm{Ho}(\mathrm{Top}))$.

Let $\mathcal{C} = (\mathcal{C}, \otimes, I, [-,-])$ be a closed monoidal category.

Definition 2.8.6. An *operad* in $\mathcal{C}$ is a collection of objects $\{\mathcal{O}(j)\}_{j \geq 0}$ in $\mathcal{C}$ such that

1. there is a right action of Σ_j on $\mathcal{O}(j)$
2. $\mathcal{O}(0) = I$
3. $I \to \mathcal{O}(1)$ exists in $\mathcal{C}$
4. composition

$$\mathcal{O}(k) \otimes \mathcal{O}(j_1) \otimes \cdots \otimes \mathcal{O}(j_k) \xrightarrow{\gamma} \mathcal{O}(j_1 + \ldots + j_k)$$

for all $k \geq 0$ and $j_1, \ldots, j_k \geq 0$ such that they are equivariant, unital, and associative.

We think about $\mathcal{O}(j)$ as an abstract way to compose j-ary operations.

Example 2.8.7. We let Assoc be the operad defined by

$$\mathrm{Assoc}(j) = \coprod_{\sigma \in \Sigma_j} I.$$

We can define $\mathrm{Comm}(j) = I$.

Example 2.8.8. If $X \in \mathcal{C}$, the *endomorphism operad* is given by

$$\mathrm{End}_X(j) = [X^{\otimes j}, X].$$

Definition 2.8.9. A *morphism of operads* $\mathcal{O} \to \mathcal{O}'$ is a sequence of maps $\psi_j : \mathcal{O}(j) \to \mathcal{O}'(j)$ for $g \geq 0$ that are equivariant, associative, and unital.

Definition 2.8.10. Given $\mathcal{O}$ an operad in $\mathcal{C}$, an $\mathcal{O}$-algebra (X, θ) in $\mathcal{C}$ is $X \in \mathcal{C}$ together with a morphism of operads $\theta : \mathcal{O} \to \mathrm{End}_X$, sending $\mathcal{O}(j) \to \mathrm{End}_X(j)$. By adjointness, we think about this as $\mathcal{O}(j) \otimes X^{\otimes j} \to X$ which are associative and unital.

This gives us a category of $\mathcal{O}$-algebras, denoted $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$.

Example 2.8.11. We have that

$$\mathrm{Alg}_{\mathrm{Assoc}}(\mathcal{C}) \cong \mathrm{Alg}(\mathcal{C})$$
$$\mathrm{Alg}_{\mathrm{Comm}}(\mathcal{C}) \cong \mathrm{CAlg}(\mathcal{C}).$$

We have that $\mathcal{M}$ is a monoidal model category if $\mathcal{O}$ is nice enough, i.e. we get an adjunction

$$\mathcal{M} \rightleftarrows \mathrm{Alg}_{\mathcal{O}}(\mathcal{M}).$$

Definition 2.8.12. A *monad* in $\mathcal{C}$ is an algebra in $(\mathrm{Fun}(\mathcal{C},\mathcal{C}),\circ,\mathrm{id}_{\mathcal{C}})$. That is, $M \in \mathrm{Alg}(\mathrm{Fun}(\mathcal{C},\mathcal{C}))$ if we have $M : \mathcal{C} \to \mathcal{C}$ together with $\mu : M \circ M \Rightarrow M$, and $\eta : \mathrm{id}_{\mathcal{C}} \Rightarrow \mathcal{C}$ that are associative and unital.

Example 2.8.13. Every adjunction $L : \mathcal{C} \rightleftarrows \mathcal{D} : R$ defines a monad RL.

Definition 2.8.14. An *algebra* (X,θ) over a monad (M,μ,η) in $\mathcal{C}$ is $X \in \mathcal{C}$ together with maps $\theta : M(X) \to X$ such that they are associative and unital, meaning that the diagrams commute:

$$\begin{tikzcd} X \arrow[r, "\eta"] \arrow[dr, equal] & M(X) \arrow[d, "\theta"] \\ & X \end{tikzcd} \qquad \begin{tikzcd} M(M(X)) \arrow[r, "\mu_{MX}"] \arrow[d, "M(\theta)"'] & M(X) \arrow[d, "\theta"] \\ M(X) \arrow[r, "\theta"'] & X. \end{tikzcd}$$

Definition 2.8.15. If M is a monad, a *morphism of M-algebras* $(X,\theta) \to (X',\theta')$ is a map $f : X \to X'$ in $\mathcal{C}$ so that the diagram commutes

$$\begin{tikzcd} MX \arrow[r, "\theta"] \arrow[d, "Mf"'] & X \arrow[d, "f"] \\ MX' \arrow[r, "\theta'"'] & X'. \end{tikzcd}$$

Example 2.8.16. Consider R a commutative ring, and the adjunction

$$- \otimes_{\mathbb{Z}} R : \mathsf{Ab} \rightleftarrows \mathrm{Mod}_R : U.$$

This forms a monad $M := - \otimes_{\mathbb{Z}} R : \mathsf{Ab} \to \mathsf{Ab}$. Then $\mathrm{Alg}_M(\mathsf{Ab})$ is equivalent to Mod_R.

This is not always true! When this happens we say the adjunction is *monadic*.
Given a monadic adjunction

$$\mathcal{C} \rightleftarrows \mathcal{D} = \mathrm{Alg}_{RL}(\mathcal{C}),$$

we get a ton of things for free:

- R will preserve colimits if RL does
- get things like free monadic resolutions, bar constructions, etc.

[Some of these notes were typed from grad school, a bit outdated]
Given an operad $\mathcal{O}$ in a nice enough monoidal category $\mathcal{C}$, we obtain a monadic adjunction:

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \arrow[l, shift left] \end{tikzcd}$$

The left adjoint provides the free $\mathcal{O}$-algebra functor, which is given on an object $X \in \mathcal{C}$ by the coequalizer in $\mathcal{C}$:

$$\coprod_{j\geq 0} \mathcal{O}(j) \otimes X^{\otimes j} \rightrightarrows \coprod_{j \geq 0} \mathcal{O}(j) \otimes_{I[\Sigma_j]} X^{\otimes j}$$

First map is induced supposing we have a canonical map $I \to X$ in $\mathcal{C}$, and the other map is induced by composition γ on $\mathcal{O}(j)$ with $j-1$-copies of $\mathcal{O}(1)$ and 1-copy of $\mathcal{O}(0)$ and thus lands to $\mathcal{O}(j-1)$. [Make this more precise]

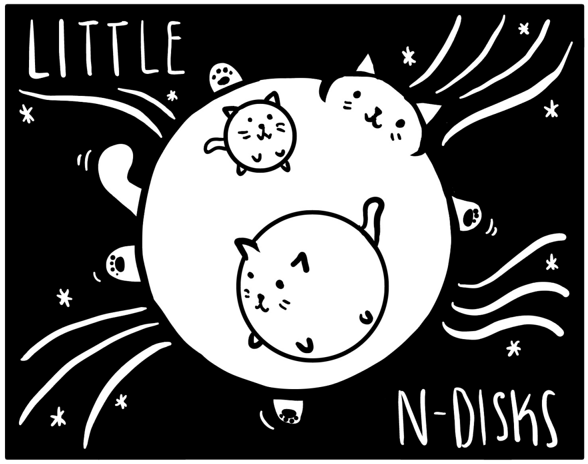
This adjunction gives defines a monad $\mathbb{O}\colon \mathcal{C} \to \mathcal{C}$. And this is always monadic (exercise). So $\mathcal{O}$-algebras in $\mathcal{C}$ are equivalent to $\mathbb{O}$-algebra in $\mathcal{C}$.

We define now an operad on the symmetric monoidal category $(\mathbf{Top},\times,*)$, where by spaces we mean topological weak Hausdorff k-spaces.

Definition 2.8.17. Let J^n be the interior of the n-dimensional unit cube $[0,1]^n$. A *little n-cube* is a rectilinear map $c : J^n \hookrightarrow J^n$. Algebraically, this means the map is of the form :

$$(t_1, \ldots, t_n) \longmapsto (a_1 + (b_1 - a_1)t_1, \ldots, a_n + (b_n - a_n)t_n),$$

with $(a_1, \ldots, a_n), (b_1, \ldots, b_n) \in [0,1]^n$ such that $0 \leq a_i \leq b_i \leq 1$, for all $1 \leq i \leq n$. The image of c defines a n-dimensional cube in $[0,1]^n$ with a non-empty interior and faces parallel to the faces of the ambient unit cube.



Definition 2.8.18. The little n-cube operad C_n is defined as follows :

$$C_n(j) = \left\{(c_1, \ldots, c_j) \mid c_i \text{ are little } n\text{-cubes with disjoint interior}\right\} \subseteq \mathrm{Map}\left(\coprod_{i=1}^{j} J^n, J^n\right).$$

The identity is defined by the element $\mathrm{id}_{J^n} \in C_n(1)$. The symmetric group Σ_j acts (freely) by permutation on the indices of the tuple $(c_1, \ldots, c_j)$. If we write $\underline{c} = (c_1, \ldots, c_j)$, we define the composition operation γ as follows :

$$\begin{array}{rcl} \gamma \colon C_n(k) \times C_n(j_1) \times \cdots \times C_n(j_k) & \longrightarrow & C_n(j_1 + \cdots + j_k) \\ (\underline{c}, \underline{d}_1, \ldots, \underline{d}_k) & \longmapsto & \underline{c} \circ (\underline{d}_1 + \cdots + \underline{d}_k). \end{array}$$

Notice that there are natural inclusions:

$$\begin{array}{ccc} C_n(j) & \hookrightarrow & C_{n+1}(j) \\ \underline{c} & \longmapsto & (c_1 \times \mathrm{id}_J, \ldots, c_j \times \mathrm{id}_J), \end{array}$$

allowing to define $C_\infty(j) = \mathrm{colim}_n C_n(j)$ for each $j \geq 0$. The composition γ extends naturally so that C_∞ is an operad.

We can reinterpret the spaces $C_n(j)$ in terms of configuration space. Let M be a n-manifold, the j-th configuration space of M is :

$$F(M; j) = \left\{(x_1, \ldots, x_j) \in M^{\times j} \mid x_r \neq x_s \text{ if } r \neq s\right\} \subseteq M^{\times j}.$$

It is a nj-manifold with Σ_j free-action on coordinates. For $1 \leq n \leq \infty$, the spaces $C_n(j)$ are Σ_j-equivariantly homotopic to $F(\mathbb{R}^n; j)$ via the map :

$$\begin{array}{rcl} C_n(j) & \longrightarrow & F(J^n; j) \\ (c_1, \ldots, c_j) & \longmapsto & (c_1(p), \ldots, c_j(p)), \end{array}$$

where $p = (\frac{1}{2}, \ldots, \frac{1}{2})$ in J^n. This makes C_1 an $\mathbb{A}_\infty$-operad, C_∞ a $\mathbb{E}_\infty$-operad, C_n a locally $(n-2)$-connected Σ-free operad.

Proposition 2.8.19. Given a pointed space X, its n-th iterated loop space $\Omega^n X$ has a natural C_n-algebra structure.

Proof. Regard $\Omega^n X$ as the space $\mathrm{Map}\left((\frac{[0,1]^n}{\partial[0,1]^n}, *), (X, *)\right)$. Define the action :

$$\theta \colon C_n(j) \times (\Omega^n X)^j \longrightarrow \Omega^n X,$$

as follows: given $(c_1, \ldots, c_j)$ in $C_n(j)$ and $(y_1, \ldots, y_j)$ in $(\Omega^n X)^j$ define $\theta(\underline{c}, \underline{y})$ as:

$$\begin{array}{ccl} \dfrac{[0,1]^n}{\partial[0,1]^n} & \longrightarrow & X \\ t & \longmapsto & \begin{cases} y_r \circ c_r^{-1}(t), & \text{if } t \in \mathrm{im}(c_r) \\ *, & \text{if } t \notin \mathrm{im}(c_r) \text{ for any } 1 \le r \le j \end{cases} \end{array}$$

One can check that all the desired diagrams commute. ◻

Recall that given a pointed space X, the associated monad of C_n is defined as:

$$C_n(X) = \left(\coprod_{j \ge 0} C_n(j) \times_{\Sigma_j} X^j \right) \Big/ \sim .$$

The above result implies that $\Omega^n X$ is also a C_n-algebra, hence there is a map $C_n(\Omega^n X) \to \Omega^n X$, for any pointed space X. There is a natural map :

$$\alpha_n : C_n(X) \longrightarrow \Omega^n \Sigma^n X,$$

defined as follows. The identity map on $\Sigma^n X$ has an adjoint $X \to \Omega^n \Sigma^n X$. Applying the functor C_n we get the left map in the composite :

$$C_n(X) \longrightarrow C_n(\Omega^n \Sigma^n X) \longrightarrow \Omega^n \Sigma^n X,$$

and the right map is defined by the C_n-algebra structure on $\Omega^n \Sigma^n X$. The above composite defines the map α_n. It is a morphism of monads, where the monad structure on the functor $\Omega^n \Sigma^n : \mathbf{Top}_* \to \mathbf{Top}_*$ is defined for any pointed space Y :

$$\Omega^n \Sigma^n \Omega^n \Sigma^n Y \longrightarrow \Omega^n \Sigma^n Y,$$

by a map $\Sigma^n \Omega^n \Sigma^n Y \to \Sigma^n Y$ which is the adjoint of the identity map $\Omega^n \Sigma^n Y \to \Omega^n \Sigma^n Y$. More concretly, the map $\alpha_n : C_n(X) \to \Omega^n \Sigma^n X$ can be regarded as follows :

$$\begin{array}{ccl} C_n(X) & \longrightarrow & \Omega^n \Sigma^n X = \mathrm{Map}\left((\dfrac{[0,1]^n}{\partial[0,1]^n}, *), (\Sigma^n X, *) \right) \\ ((c_1, \ldots, c_j), (x_1, \ldots, x_j)) & \longmapsto & \left(\begin{array}{ccl} \frac{[0,1]^n}{\partial[0,1]^n} & \longrightarrow & \Sigma^n X \\ t & \longmapsto & \begin{cases} t \in \frac{[0,1]^n}{\partial[0,1]^n} = S^n = \Sigma^n\{*, x_i\}, \text{if } t \in \mathrm{im}(c_i) \subseteq J^n \\ *, \ \text{if } t \notin \mathrm{im}(c_i) \text{ for any } 1 \le i \le j \end{cases} \end{array} \right). \end{array}$$

Theorem 2.8.20 (Approximation)**.** For any based space X, there is a natural map of C_n-algebras :

$$\alpha_n : C_n(X) \to \Omega^n \Sigma^n X,$$

for $1 \le n \le \infty$, and α_n is a weak homotopy equivalence if X is connected.

Proof. We construct the following commutative diagram :

$$\begin{tikzcd}
C_n(X) \arrow[r, hook] \arrow[d] & \widetilde{X}_n \arrow[r, "\widetilde{p}_n"] \arrow[d] & C_{n-1}(\Sigma X) \arrow[d] \\
\Omega^n\Sigma^n X \arrow[r, hook] & P\Omega^{n-1}\Sigma^n X \arrow[r, "p"] & \Omega^{n-1}\Sigma^n X,
\end{tikzcd}$$

where p is the usual path fibration to a space with fiber its loop space. The space $\widetilde{X}_n$ is constructed such that it is contractible and $\widetilde{p}_n$ is a quasifibration if X is connected. □

Theorem 2.8.21 (Recognition)**.** If X is a connected grouplike C_n-algebra, there exists a based space Y and a weak equivalence of C_n-algebras between $\Omega^n Y$ and X.

In order to construct this delooping of X, we use the two-sided bar construction in $\mathbf{Top}_*$. Given a monad (M, μ, η) in $\mathcal{E}$ and a category $\mathcal{C}$, a M-functor in $\mathcal{C}$ is a functor $F : \mathcal{E} \to \mathcal{C}$ with a natural transformation $\lambda : FM \Rightarrow F$ such that the following diagram commutes :

$$\begin{tikzcd}
F(M(M(X))) \arrow[r, "F(\mu_X)"] \arrow[d, "\lambda_{M(X)}"'] & FM(X), \arrow[d, "\lambda_X"] \\
FM(X) \arrow[r, "\lambda_X"] & F(X),
\end{tikzcd}
\qquad
\begin{tikzcd}
F(X) \arrow[r, "F(\eta_X)"] \arrow[dr, equal] & F(M(X)) \arrow[d, "\lambda_X"] \\
 & F(X).
\end{tikzcd}$$

For instance, (M, μ) is itself a M-functor in $\mathcal{E}$.

Definition 2.8.22. Given a monad (M, μ, η) in $\mathcal{E}$, a M-functor (F, λ) in $\mathcal{C}$, and a M-algebra (X, ξ) in $\mathcal{E}$, define the *two-sided bar construction of* (F, M, X) by :

$$B_q(F, M, X) = F(M^q(X)).$$

The object is simplicial in $\mathcal{C}$:

$$F(X) \leftleftarrows F(M(X)) \;\; F(M(M(X))) \;\; F(M(M(M(X)))) \qquad \cdots$$

where the blue arrows are induced by $\xi : M(X) \to X$, the red arrows by $\lambda : F(M(X)) \to F(X)$, the green arrows by $\mu : M(M(X)) \to M(X)$, and the black arrows by $\eta : X \to M(X)$. We denote its geometric realization by $B(F, M, X) =\mid B_*(F, M, X) \mid$.

Proof. The operad C_n is replaced by a "nicer" equivalent operad D so that $B_*(F, D, X)$ is a strictly proper simplicial space. We construct a zig-zag of maps :

$$X \longleftarrow B(D, D, X) \longrightarrow B(\Omega^n\Sigma^n, D, X) \longrightarrow \Omega B(\Sigma^n, D, X).$$

The map $B(D, D, X) \to X$ is induced by $D(X) \to X$ as X is a D-algebra and $B(D, D, X)$ should be regarded as the usual simplicial resolution of X. The map $B(D, D, X) \to B(\Omega^n\Sigma^n, D, X)$ is induced by $\alpha_n : D \to \Omega^n\Sigma^n$ (and should now be regarded as a morphism of D-functors). It is a weak equivalence when X is connected (not obvious on the simplicial resolution). The last map $B(\Omega^n\Sigma^n, D, X) \to \Omega^n B(\Sigma^n, D, X)$ should be regarded as the non-trivial weak equivalence $\mid \Omega X_* \mid \to \Omega \mid X_* \mid$, true only when X is connected. Thus let Y be $B(\Sigma^n, D, X)$. □

Chapter 3

Higher categories

3.1 Foundations

Definition 3.1.1. A simplicial set $\mathcal{C}$ is an ∞-*category* (or *quasi-category*) if it has inner horn filling — for all $0 < k < n$, we have

$$\begin{tikzcd} \Lambda^k_n \arrow[r] \arrow[d] & \mathcal{C} \\ \Delta^n \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

We shall see that ∞-categories are fibrant objects in sSet with the Joyal model structure.

Example 3.1.2.

1. If $\mathcal{C}$ is a Kan complex, then it is an ∞-category
2. If $\mathcal{C}$ is a category, then $N\mathcal{C}$ is an ∞-category.

Definition 3.1.3. Given an ∞-category $\mathcal{C}$, the *objects* of $\mathcal{C}$ are the vertices,[1] the *morphisms* are 1-simplices. We have *source* and *target* maps $d^1, d^0 : \mathcal{C}_1 \to \mathcal{C}_0$.[2] We define the *set of morphisms* from X to Y as the pullback

$$\begin{tikzcd} \hom_{\mathcal{C}}(X,Y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{C}_1 \arrow[d, "{(s,t)}"] \\ \mathcal{C}_1 \arrow[r, "{(X,Y)}"'] & \mathcal{C}_0 \times \mathcal{C}_0. \end{tikzcd}$$

[1] $X \in \mathcal{C}$ means $X \in \mathcal{C}_0$

[2] We write $f : X \to Y$ in $\mathcal{C}$ to mean $f \in \mathcal{C}_1$ with $s(f) = X$ and $t(f) = Y$.

We have that $\hom_{\mathcal{C}}(X,Y)$ is the set of vertices of a simplicial set $\mathrm{Hom}_{\mathcal{C}}(X,Y)$, which forms a Kan complex that we define later.

Definition 3.1.4. Given $X \in \mathcal{C}$ we define $\mathrm{id}_X \in \mathcal{C}_1$ by $s^0(X)$.

How do we compose? Composition won't be unique, but it will be unique *up to homotopy.*

Given $f : X \to Y$ and $g : Y \to Z$ in $\mathcal{C}$, this determines a map of simplicial sets $\Lambda^2_1 \to \mathcal{C}$. By inner horn lifting, we have

$$\begin{tikzcd} \Lambda^2_1 \arrow[r] \arrow[d] & \mathcal{C} \\ \Delta^2 \arrow[ur, dashed] & \end{tikzcd}$$

We refer to the filling as a *composition*:

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "h"'] & & Z. \end{tikzcd}$$

Exercise 3.1.5. Given an ∞-category $\mathcal{C}$, how can we define $\mathcal{C}^{\mathsf{op}}$? Would want that $N(\mathcal{C}^{\mathsf{op}}) \cong (N\mathcal{C})^{\mathsf{op}}$.[3]

Detour: Let $A \in \mathsf{Cat}$, and let $\mathcal{C}$ be a cocomplete category. Recall that $\mathrm{Fun}(A^{\mathsf{op}}, \mathsf{Set})$ is the free cocompletion. Given a functor $A \to \mathcal{C}$, by universal property there is a map

$$\begin{tikzcd} A \arrow[r, "Q"] \arrow[d, hook] & \mathcal{C} \\ \mathrm{Fun}(A^{\mathsf{op}}, \mathsf{Set}) \arrow[ur, dashed, "|-|_Q"'] & \end{tikzcd}$$

This gives us an adjunction

$$|-|_Q : \mathrm{Fun}(A^{\mathsf{op}}, \mathsf{Set}) \rightleftarrows \mathcal{C} : \mathrm{Sing}_Q(-).$$

Here $\mathrm{Sing}_Q(-) = \mathrm{Hom}_{\mathcal{C}}(Q(-), X)$.

Example 3.1.6. If $\mathcal{C} = \mathrm{Top}$, then we can take $\Delta_{\mathrm{Top}} : \Delta \to \mathrm{Top}$, sending $[n]$ to Δ^n_{Top}. In this case, we recover the usual $|-|$ and $\mathrm{Sing}(-)$ adjunction.

Example 3.1.7. If $\mathcal{C} = \mathsf{Cat}$, there is a functor $\Delta \to \mathsf{Cat}$ sending $[n]$ to the associated poset category. We get an associated adjunction:

$$\tau : \mathrm{sSet} \rightleftarrows \mathsf{Cat} : N,$$

since $N = \mathrm{Hom}_{\mathsf{Cat}}([-], \mathcal{C})$.

Exercise 3.1.8. Describe $\tau : \mathrm{sSet} \to \mathsf{Cat}$ explicitly.

We call τ the fundamental category functor, essentially it will produce the homotopy category of an ∞-category.

Definition 3.1.9. Given an ∞-category $\mathcal{C}$, two morphisms $f : X \to Y$ and $g : Y \to Z$ are *homotopic*, written $f \simeq g$, if there exists a 2-simplex $\sigma : \Delta^2 \to \mathcal{C}$ with boundary (g, f, id_X):

$$\begin{tikzcd} & X \arrow[dr, "g"] & \\ X \arrow[ur, "\mathrm{id}_X"] \arrow[rr, "f"'] & & Y. \end{tikzcd}$$

[3]Every Kan complex has that $\mathcal{C}^{\mathsf{op}} \cong \mathcal{C}$.

Example 3.1.10. If $\mathcal{C}$ is an ordinary category, then in $N\mathcal{C}$, we have that $f \simeq g$ if and only if $f = g$.

Proposition 3.1.11. Given $\mathcal{C}$ an ∞-category, and $X, Y \in \mathcal{C}$, the homotopy relation provides an equivalence relation on $\hom_{\mathcal{C}}(X, Y)$.

Definition 3.1.12. We denote by $[f]$ the homotopy class of f.

Sketch. We first need to show reflexivity, so we want to find a 2-cell witnessing

$$\begin{tikzcd} & X \arrow[dr, "f"] & \\ X \arrow[ur, equal] \arrow[rr, "f"'] & & Y. \end{tikzcd}$$

We check that this is $s_0(f)$, where $f \in \mathcal{C}_1$, and $s_0 : \mathcal{C}_1 \to \mathcal{C}_2$.

For symmetry, suppose we have $f \simeq g$. We want to show $g \simeq f$. We can fill a Λ^3_2 witnessing this.

Transitivity is left as an exercise. ◻

Definition 3.1.13. Given $\mathcal{C}$ an ∞-category, define the 1-category $\mathrm{Ho}(\mathcal{C})$ to be the *homotopy category*, given by

$$\mathrm{Ob}\mathrm{Ho}(\mathcal{C}) = \mathcal{C}_0$$
$$\mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(X, Y) = \hom_{\mathcal{C}}(X, Y)/\simeq .$$

In order to show this, we need to argue that composition is well-defined up to homotopy.

Suppose we have two compositions

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "h_1"'] & & Z. \end{tikzcd} \qquad \begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "h_2"'] & & Z. \end{tikzcd}$$

We want to argue that $h_1 \simeq h_2$. This can be done by filling the horn of a 3-simplex.

Proposition 3.1.14. When we restrict the adjunction $\tau \dashv N$ to ∞-categories, we get an adjunction

$$\mathrm{Ho}(-) : \mathsf{Cat}_\infty \rightleftarrows \mathsf{Cat} : N.$$

The way to compose arrows is contractible.

Definition 3.1.15. The internal hom of simplicial set is given as follows. Given X and Y simplicial sets, we define $\mathrm{Hom}_\bullet(X, Y)$ as:

$$\mathrm{Hom}_\bullet(X, Y) = \mathrm{Hom}_{\mathrm{sSet}}(\Delta^\bullet \times X, Y).$$

Theorem 3.1.16. The inclusion $\Lambda^2_1 \hookrightarrow \Delta^2$ induces a map

$$\mathrm{Hom}_*(\Delta^2, \mathcal{C}) \to \mathrm{Hom}_*(\Lambda^2_1, \mathcal{C})$$

which is a trivial Kan fibration if and onlly if $\mathcal{C}$ is an ∞-category.

Proof. Here is the main idea. We need to show there is a lifting:

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d, hook] & \mathrm{Hom}_\bullet(\Delta^2, \mathcal{C}) \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed, "\exists"] & \mathrm{Hom}_\bullet(\Lambda^2_1, \mathcal{C}). \end{tikzcd}$$

By adjunction, this is equivalent to have a lifting:

$$\begin{tikzcd} (\Delta^n \times \Lambda^2_1) \coprod_{\partial\Delta^n \times \Lambda^2_1} (\partial\Delta^n \times \Delta^2) \arrow[r] \arrow[d] & \mathcal{C} \\ \Delta^n \times \Delta^2 \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

This will follow from seeing that $\mathcal{C}$ is a fibrant object in model structure on sSet, and the left vertical map is a trivial cofibration, because it is generated by inner anodyne $\Lambda^n_i \hookrightarrow \Delta^n$ cofibrations. □

Definition 3.1.17. An inner fibration in simplicial sets is a map which has the right lifting property with respect to the inclusions $\Lambda^n_i \hookrightarrow \Delta^n$.

As a consequence, we can take a pullback diagram:

$$\begin{tikzcd} P \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Hom}_*(\Delta^2, \mathcal{C}) \arrow[d] \\ \Delta^0 \arrow[r] & \mathrm{Hom}_*(\Lambda^2_1, \mathcal{C}). \end{tikzcd}$$

Then the pullback $P \to \Delta^0$ should be a trivial fibration, meaning that P is a contractible Kan complex.

Definition 3.1.18. Given $\mathcal{C}$ an ∞-category and $X, Y \in \mathcal{C}$, recall that a map $f : X \to Y$ corresponds to $\Delta^1 \to \mathcal{C}$ whose faces are X and Y. An *n-morphism* from X to Y is simply a map $\Delta^n \to \mathcal{C}$ such that $\Delta^{\{0,\dots,n-1\}} = X$ and $\Delta^{\{n\}} = Y$.

For $n \geq 2$, all n-morphisms are invertible in some sense.

Definition 3.1.19. Two objects X and Y in $\mathcal{C}$ are *equivalent*, written $X \simeq Y$, if there exists a 1-morphism $f : X \to Y$ in $\mathcal{C}$ such that $[f]$ in $\mathrm{Ho}(\mathcal{C})$ is an *isomorphism*.

Definition 3.1.20. An ∞-*groupoid* is an ∞-category for which $\mathrm{Ho}(\mathcal{C})$ is a groupoid, meaning all the 1-morphisms are equivalences.

Theorem 3.1.21. (Homotopy hypothesis) We get that $\mathcal{C}$ is an ∞-groupoid if and only if $\mathcal{C}$ is a Kan complex.

Example 3.1.22. How to define the opposite $\mathcal{C}^{\mathsf{op}}$ of an ∞-category? This is a good exercise to try on your own first. Here is the solution. We can view Δ as a subcategory of finite linear ordered sets Lin with non-decreasing functions. This has an involution $\mathit{Lin} \to \mathit{Lin}$ which sends a poset $(I, \leq)$ to $(I, \leq^{\mathsf{op}})$ where $i \leq^{\mathsf{op}} j$ whenever $j \leq i$. This defines a similar functor $\mathsf{op}\colon \Delta \to \Delta$ which is identity on object, and sends a map $\alpha\colon [m] \to [n]$ to $\mathsf{op}(\alpha)\colon [m] \to [n]$ defined as $i \mapsto n - \alpha(m-i)$. Therefore, given a simplicial set $X_\bullet$, we can define $X^{\mathsf{op}}_\bullet$ by precomposing by the previous functor. Essentially, $d^{\mathsf{op}}_i = d_{n-i}$ and $s^{\mathsf{op}}_i = s_{n-i}$. Doing this for an ∞-category shows we switch source and target.

Proposition 3.1.23. If $\mathcal{C}$ is an ∞-category, then $\mathcal{C}^{\mathsf{op}}$ is also an ∞-category.

Proof. Notice we have an isomorphism of simplicial sets $(\Delta^n)^{\mathsf{op}} \cong \Delta^n$ and that sends $(\Lambda^n_i)^{\mathsf{op}}$ to Λ^n_{n-i}. □

3.2 Equivalence of ∞-categories

What is the correct notion of an equivalence of ∞-categories? Let us first see how the notion of opposite is compatible with ordinary sense.

Proposition 3.2.1. Given $\mathcal{C}$ an ordinary category, then we obtain an isomorphism of simplicial sets $N(\mathcal{C})^{\mathsf{op}} \cong N(\mathcal{C}^{\mathsf{op}})$

Proof. The string

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \to X_n$$

is sent to

$$X_n \xrightarrow{f_n^{\mathrm{op}}} X_{n-1} \xrightarrow{f_{n-1}^{\mathrm{op}}} \cdots \to X_0$$

□

Just as groupoids are equivalent to their opposite categories, the same should be true for ∞-groupoids. This is first observed by the following result.

Proposition 3.2.2. Given X a topological space, then $\mathrm{Sing}(X) \cong \mathrm{Sing}(X)^{\mathrm{op}}$ as simplicial sets.

Proof. A n-simplex $|\Delta^n| \to X$ is send to $|\Delta^n| \xrightarrow{\cong} |\Delta^n| \to X$ where the homeomorphism is defined via $(t_0, t_1, \dots, t_n) \mapsto (t_n, t_{n-1}, \dots, t_0)$. □

Therefore, given X a Kan complex, we obtain:

$$X \xleftarrow{\simeq} \mathrm{Sing}(|X|) \xrightarrow{\cong} \mathrm{Sing}(|X|)^{\mathrm{op}} \xrightarrow{\simeq} X^{\mathrm{op}}.$$

Of course, isomorphism of simplicial sets is too strong of a notion for an equivalence of ∞-categories. At most, we would want the notion to not be stronger on spaces: two Kan complexes that are equivalent as homotopy-types should also be equivalent as ∞-categories.

Let us get inspired by ordinary categories. Two ordinary categories $\mathcal{C}$ and $\mathcal{D}$ are equivalent if a functor $F\colon \mathcal{C} \to \mathcal{D}$ induces an isomorphism of sets $\mathrm{Hom}_{\mathcal{C}}(X, Y) \cong \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$ and $D \cong F(C)$ for all $D \in \mathcal{D}$ for some $C \in \mathcal{C}$. An easier way to generalize, is to have another functor $G\colon \mathcal{D} \to \mathcal{C}$ such that $F \circ G$ and $G \circ F$ are equivalent in the functor categories to the identity functors. Let us also record the following.

Example 3.2.3. There is a canonical model structure on Cat where weak equivalences are given by equivalences of categories, cofibrations are functors that are injective on objects, fibrations are isofibrations. An isofibration is a functor $p\colon \mathcal{C} \to \mathcal{D}$ such that for all $C \in \mathcal{C}$, for all isomorphism $g\colon D \xrightarrow{\cong} D'$ in $\mathcal{D}$ where $p(C) = D$, there exists $f\colon C \to C'$ such that $p(f) = g$.

Definition 3.2.4. A functor of ∞-categories $\mathcal{C} \to \mathcal{D}$ is a morphism of simplicial sets (i.e. a natural transformation).

This definition provides all the expectations of what a functor should do: preserve the choice of compositions, preserve equivalences, preserve identities, send n-morphisms to n-morphisms (exercise). Although evident from the definition, it is crucial to keep in mind that it is **not** enough to define a functor by simply assigning objects and 1-morphisms, we must also define on all higher morphisms.

Example 3.2.5. An ordinary functor $\mathcal{C} \to \mathcal{D}$ defines a functor $N(\mathcal{C}) \to N(\mathcal{D})$ of ∞-categories.

Example 3.2.6. Given an ∞-category $\mathcal{C}$ and an ordinary category $\mathcal{D}$, then the data of a functor $\mathcal{C} \to N(\mathcal{D})$ is equivalent to a functor $\mathrm{Ho}(\mathcal{C}) \to \mathcal{D}$.

Example 3.2.7. Given an equivalence $f \simeq g$ in $\mathcal{C}$, we obtain $F(f) \simeq F(g)$ for any functor $F\colon \mathcal{C} \to \mathcal{D}$, and thus we obtain an ordinary functor $F\colon \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{D})$.

Example 3.2.8. Given $\mathcal{C}$ an ∞-category, and X a topological space, a functor $\mathcal{C} \to \mathrm{Sing}(X)$ is equivalent to a continuous map $|\mathcal{C}| \to X$.

Definition 3.2.9. A diagram in an ∞-category is a morphism of simplicial sets $K_\bullet \to \mathcal{C}$, where $K_\bullet$ is any simplicial set.

Example 3.2.10. A diagram $\Delta^1 \times \Delta^1 \to \mathcal{C}$ makes sense of a commutative diagram:

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d] \arrow[dr] & \bullet \arrow[d] \arrow[dl, phantom, "\Swarrow"] \\
\bullet \arrow[r] \arrow[ur, phantom, "\Nearrow"] & \bullet
\end{tikzcd}$$

As hint of what the ∞-category of functors of ∞-categories, we have the following.

Example 3.2.11. We have an isomorphism of simplicial sets:

$$N(\mathrm{Fun}(\mathcal{C},\mathcal{D})) \cong \mathrm{Hom}_\bullet(N(\mathcal{C}), N(\mathcal{D})).$$

Theorem 3.2.12. Given K a simplicial set, $\mathcal{C}$ an ∞-category, then $\mathrm{Hom}_\bullet(K,\mathcal{C})$ is an ∞-category.

Proof. Notice that $\mathrm{Hom}_\bullet(K,-)$ preserves trivial Kan fibrations (because sSet with Kan model structure is a monoidal model category). Therefore, as $\mathcal{C}$ is an ∞-category, by Theorem 3.1.16, we obtain:

$$\mathrm{Hom}_\bullet(K, \mathrm{Hom}_\bullet(\Delta^2,\mathcal{C})) \longrightarrow \mathrm{Hom}_\bullet(K, \mathrm{Hom}_\bullet(\Lambda^2_1,\mathcal{C}))$$

which by symmetry, is equivalent to trivial Kan fibration:

$$\mathrm{Hom}_\bullet(\Delta^2, \mathrm{Hom}_\bullet(K,\mathcal{C})) \longrightarrow \mathrm{Hom}_\bullet(\Lambda^2_1, \mathrm{Hom}_\bullet(K,\mathcal{C}))$$

We conclude by Theorem 3.1.16 again. $\square$

Definition 3.2.13. Given an ∞-category $\mathcal{C}$, and a simplicial set K, we denote by $\mathrm{Fun}(K,\mathcal{C})$ the ∞-category $\mathrm{Hom}_\bullet(K,\mathcal{C})$.

Definition 3.2.14. A natural transformation between functors $\mathcal{C} \to \mathcal{D}$ is a morphism in $\mathrm{Fun}(\mathcal{C},\mathcal{D})$, i.e. a map of simplicial sets $\Delta^1 \times \mathcal{C} \to \mathcal{D}$.

Definition 3.2.15. Given $\mathcal{C}$ an ∞-category, define $\mathcal{C}^{\simeq}$ to be the maximal ∞-groupoid of $\mathcal{C}$: the subsimplicial set for which n-simplices carry edges to equivalences in $\mathcal{C}$.

Example 3.2.16. For $\mathcal{C}$ an ordinary category, we have an isomorphism of simplcial sets $N(\mathcal{C}^{\cong}) \cong N(\mathcal{C})^{\simeq}$.

Exercise: Show $\mathcal{C}^{\simeq}$ is indeed a Kan complex.

Definition 3.2.17. The homotopy category of ∞-categories $hQCat$ is the category for which objects are ∞-categories and for which the hom sets are the equivalence classes of functors:

$$\mathrm{Hom}_{hQCat}(\mathcal{C},\mathcal{D}) = \pi_0(\mathrm{Fun}(\mathcal{C},\mathcal{D})^{\simeq}).$$

We obtain an adjunction:

$$\begin{tikzcd} hTop \arrow[r, hook, shift left] \arrow[r, phantom, "\perp"] & hQCat. \arrow[l, shift left, "(-)^{\simeq}"] \end{tikzcd}$$

Definition 3.2.18. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is an equivalence of ∞-categories if it is an isomorphism in $hQCat$.

Example 3.2.19. Let $\mathcal{C}$ and $\mathcal{D}$ be ordinary categories. A functor $\mathcal{C} \to \mathcal{D}$ is an equivalence of categories if and only if $N(\mathcal{C}) \to N(\mathcal{D})$ is an equivalence of ∞-categories.

Example 3.2.20. Given X and Y are Kan complexes, then $X \to Y$ is a simplicial homotopy equivalence if and only if it is an equivalence of ∞-categories.

Remark 3.2.21. Given $\mathcal{C}$ and $\mathcal{D}$ are ∞-categories, if $\mathcal{C} \to \mathcal{D}$ is an equivalence of ∞-categories, then it is a simplicial homotopy equivalence. However, the converse is not true.

Example 3.2.22. If $\mathcal{C} \to \mathcal{D}$ is an equivalence of ∞-categories, where $\mathcal{D}$ is actually a Kan complex, then $\mathcal{C}$ is also a Kan complex.

Definition 3.2.23. The Joyal model structure on sSet can be defined as follows. The fibrant objects are ∞-categories, the weak equivalences on fibrant objects are precisely the equivalence of ∞-categories, cofibrations are monomorphisms, fibrations are isofibrations (inner fibrations with identical property than ordinary case). We obtain a Quillen adjunction between the two model structures:

$$\begin{tikzcd} \mathrm{sSet}_{\mathrm{Joyal}} \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathrm{sSet}_{\mathrm{Kan}}. \arrow[l, shift left] \end{tikzcd}$$

Last time: Recall that a 1-morphism in $\mathrm{Fun}(\mathcal{C},\mathcal{D})$[4] is precisely a natural transformation $\eta : F \to G$, where $F, G : \mathcal{C} \to \mathcal{D}$. In other words, it is $\eta : \Delta^1 \times \mathcal{C} \to \mathcal{D}$.

We have hQCat = Ho(Cat_∞), where objects are infinity categories, and the morphisms are

$$\mathrm{Hom}_{\mathrm{hQCat}}(\mathcal{C},\mathcal{D}) = \pi_0\left(\mathrm{Fun}(\mathcal{C},\mathcal{D})^{\simeq}\right).$$

That is, it is the set of equivalence classes of functors $\mathcal{C} \to \mathcal{D}$.

If $\mathcal{C}$ is an ∞-category, and $X, Y \in \mathcal{C}$, we defined $\mathrm{Hom}_{\mathcal{C}}(X,Y)_\bullet$ to be the simplicial set given by the pullback

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(X,Y)_\bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}(\Delta^1,\mathcal{C}) \arrow[d] \\
\Delta^0 \arrow[r] & \mathrm{Fun}(\{0\},\mathcal{C})_\bullet \times \mathrm{Fun}(\{1\},\mathcal{C}).
\end{tikzcd}$$

Proposition 3.2.24. We have that $\mathrm{Hom}_{\mathcal{C}}(X,Y) \in \mathrm{Kan}$.

Sketch. This follows from a more general fact that for $A \hookrightarrow B$ a subsimplicial set with $A_0 = B_0$, and $\mathcal{C}$ an ∞-category, then P is always a Kan complex

$$\begin{tikzcd}
P \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}(B,\mathcal{C}) \arrow[d] \\
\Delta^0 \arrow[r, "f"'] & \mathrm{Fun}(A,\mathcal{C}).
\end{tikzcd}$$

Need to show that every u in $\mathrm{Fun}(B,\mathcal{C})_1$ *in the pullback* is a weak equivalence. We have an evaluation map for every $b \in B_0 = A_0$, given by $\mathrm{ev}_b : \mathrm{Fun}(B,\mathcal{C}) \to \mathrm{Fun}(\{b\},\mathcal{C})$, mapping u to $u_{f(b)}$. We claim that $u_{f(b)} = \mathrm{id}_{f(b)}$, since the diagram commutes

$$\begin{tikzcd}
\mathrm{Fun}(B,\mathcal{C}) \arrow[rr] \arrow[dr] & & \mathrm{Fun}(\{b\},\mathcal{C}) \\
& \mathrm{Fun}(A,\mathcal{C}) \arrow[ur] &
\end{tikzcd}$$

□

Example 3.2.25. Given $f \simeq g$ in an ∞-category $\mathcal{C}$, they must belong in same path component of $\mathrm{Hom}_{\mathcal{C}}(X,Y)$, and so $\mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(X,Y) \cong \pi_0(\mathrm{Hom}_{\mathcal{C}}(X,Y))$.

Theorem 3.2.26. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is an equivalence of ∞-categories if and only if we have both:

- weak homotopy equivalence $\mathrm{Hom}_{\mathcal{C}}(X,Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y))$ for all objects $X, Y \in \mathcal{C}$;
- $\pi_0(\mathcal{C}^{\simeq}) \to \pi_0(\mathcal{D}^{\simeq})$ is surjective.

3.3 Adjoint functors

Definition 3.3.1. Let $F : \mathcal{C} \to \mathcal{D}$, and $G : \mathcal{D} \to \mathcal{C}$ be functors of ∞-categories. We say that $F \dashv G$ if there exist natural transformations $\eta : \mathrm{id}_{\mathcal{C}} \to GF$ ad $\epsilon : FG \to \mathrm{id}_{\mathcal{D}}$ so that:

1. there exists $\Delta^2 \to \mathrm{Fun}(\mathcal{C},\mathcal{D})$ witnessing

$$\begin{tikzcd}
& FGF \arrow[dr, "\epsilon \mathrm{id}_F"] & \\
F\mathrm{id}_{\mathcal{C}} \arrow[ur, "\mathrm{id}_{\mathcal{C}}\eta"] \arrow[rr, "\mathrm{id}"'] & & \mathrm{id}_{\mathcal{C}}F.
\end{tikzcd}$$

[4]The simplicial set $\mathrm{Fun}(\Delta^\bullet \times \mathcal{C},\mathcal{D})$

2. there exists $\Delta^2 \to \mathrm{Fun}(\mathcal{D},\mathcal{C})$ witnessing

$$\begin{tikzcd} & GFG \arrow[dr, "\mathrm{id}\epsilon"] & \\ \mathrm{id}_{\mathcal{C}}G \arrow[ur, "\eta\mathrm{id}"] \arrow[rr, "\mathrm{id}"'] & & G\mathrm{id}_{\mathcal{C}}. \end{tikzcd}$$

Remark 3.3.2. We have that $\eta : \mathrm{id} \to GF$ depends only on $[\eta]$ in $\mathrm{Ho}(\mathrm{Fun}(\mathcal{C},\mathcal{D}))$. If η is given, then ϵ is unique up to homotopy.

Example 3.3.3. If $\mathcal{C}$ and $\mathcal{D}$ are ordinary categories, then we have a 1-categorical adjunction

$$F : \mathcal{C} \rightleftarrows \mathcal{D} : G$$

if and only if we have an ∞-categorical adjunction

$$NF : N\mathcal{C} \rightleftarrows N\mathcal{D} : NG.$$

Example 3.3.4. If $X, Y \in \mathrm{Kan}$, then $F : X \to Y$ is an adjoint if and only if F is a homotopy equivalence of simplicial sets. The unit and counit become the witnesses of homotopy equivalence.

Remark 3.3.5. If we have an adjunction $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ of ∞-categories, then F and G are homotopy equivalences of simplicial sets. The converse is not true in general.

Exercise 3.3.6. If $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of ∞-categories, then it is both a left and right adjoint functor.

Proposition 3.3.7. Given $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ of ∞-categories, then

$$\mathrm{Ho}(F) : \mathrm{Ho}(\mathcal{C}) \rightleftarrows \mathrm{Ho}(\mathcal{D}) : \mathrm{Ho}(G)$$

is an adjunction of 1-categories. That is, **if** we know $F \dashv G$ in ∞-categories, then to check if $\eta : \mathrm{id}_{\mathcal{C}} \to GF$ is a unit, it is enough to check that $\mathrm{Ho}(\eta)$ is the unit.

However the converse is not true!

Warning: Suppose we take $F : \Delta^0 \to X$ with $X \in \mathrm{Kan}$ simply connected, and F picks $x \in X_0$. Then $\mathrm{Ho}(F) \dashv \mathrm{Ho}(G)$ because $\mathrm{Ho}(X)$ will be simply connected. But it does not imply that $F \dashv G$ unless X is contractible.

There $\mathrm{Hom}_{\mathrm{Ho}(\mathcal{D})}(FC, D) \cong \mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(C, GD)$ for any $C \in \mathcal{C}$ and $D \in \mathcal{D}$.

Theorem 3.3.8. Take $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ functors of ∞-categories. Then $F \dashv G$ with unit η if and only if the composite

$$\mathrm{Hom}_{\mathcal{D}}(FC, D) \xrightarrow{G} \mathrm{Hom}_{\mathcal{C}}(GFC, GD) \xrightarrow{\eta^*} \mathrm{Hom}_{\mathcal{C}}(C, GD)$$

is a weak homotopy equivalence between Kan complexes (aka a homotopy equivalence) for all C, D.

The forward direction is straightforward, but the backwards direction uses (co)cartesian fibration stuff.

3.4 Limits and colimits

Recall that if $\mathcal{C}$ is an ordinary category, then $i \in \mathcal{C}$ is *initial* if for all $X \in \mathcal{C}$, there is a unique $i \xrightarrow{!} X$. That is, $\mathrm{Hom}_{\mathcal{C}}(i, X) = *$.

Definition 3.4.1. In an ∞-category $\mathcal{C}$, we have that $i \in \mathcal{C}$ is *initial* if $\mathrm{Hom}_{\mathcal{C}}(i, X) \simeq *$ is contractible for all $X \in \mathcal{C}$.

Definition 3.4.2. Let $\mathcal{C}$ be an ∞-category, and $K_\bullet \in \mathrm{sSet}$. Then for any $X \in \mathcal{C}$, denote by $\underline{X} \in \mathrm{Fun}(K, \mathcal{C})$ the constant functor valued at X. The assignment $X \mapsto \underline{X}$ defines a diagonal map

$$\Delta : \mathcal{C} \to \mathrm{Fun}(K, \mathcal{C}).$$

This is defined by precomposing with $K \to \Delta^0$, and looking at $\mathcal{C} \simeq \mathrm{Fun}(\Delta^0, \mathcal{C}) \to \mathrm{Fun}(K, \mathcal{C})$.

Definition 3.4.3. Let $u : K \to \mathcal{C}$ be a diagram. We say a natural transformation $\alpha : \underline{L} \to u$ exhibits $L \in \mathcal{C}$ as *a limit of* u if for all $X \in \mathcal{C}$, we have that the composite

$$\mathrm{Hom}_{\mathcal{C}}(X, L) \xrightarrow{\Delta} \mathrm{Hom}_{\mathrm{Fun}(K,\mathcal{C})}(\underline{X}, \underline{L}) \xrightarrow{\alpha_*} \mathrm{Hom}_{\mathrm{Fun}(K,\mathcal{C})}(\underline{X}, u)$$

is a (weak) homotopy equivalence of Kan complexes.

Definition 3.4.4. We say that $\beta : u \to \underline{C}$ exhibits C as a *colimit of* u if, for all $Y \in \mathcal{C}$, the composite

$$\mathrm{Hom}_{\mathcal{C}}(C, Y) \xrightarrow{\Delta} \mathrm{Hom}_{\mathrm{Fun}(K,\mathcal{C})}(\underline{C}, \underline{Y}) \xrightarrow{\beta^*} \mathrm{Hom}_{\mathrm{Fun}(K,\mathcal{C})}(u, \underline{C})$$

is a (weak) homotopy equivalence.

Note that if α or β exist, they are unique up to equivalence.

Example 3.4.5. If $\mathcal{C}$ is an ordinary category, then $u : K \to N\mathcal{C}$ is equivalent to a map $\tau(u) : \tau K \to \mathcal{C}$. We can check that $L \in \mathcal{C}$ is $\lim(\tau u)$ in a 1-categorical sense if and only if $L \in \mathcal{C}$ is a limit of u in an ∞-categorical sense.

Example 3.4.6. Let $f : X \to Y$ in an ∞-cat $\mathcal{C}$. Then f is an equivalence if and only if f exhibits Y as a colimit $\{X\} \to \mathcal{C}$, if and only if f exhibits X as a limit $\{Y\} \to \mathcal{C}$.

Example 3.4.7. Taking the identity diagram $\emptyset \to \mathcal{C}$, the notion of limit/colimit matches the notion of terminal/initial object.

Proposition 3.4.8. A limit $L \in \mathcal{C}$ is unique up to homotopy. Therefore we usually define it as $\lim_K(u)$.

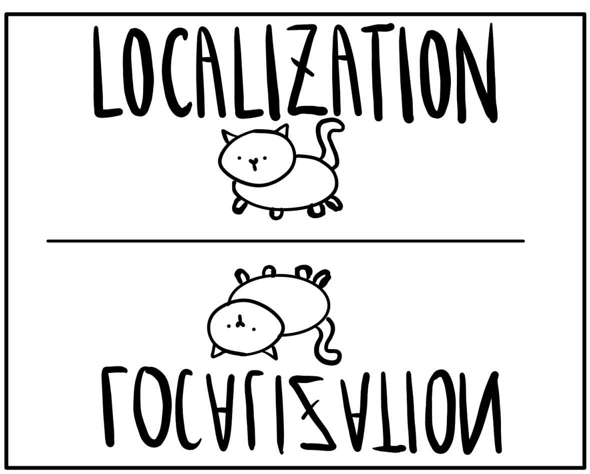
Proposition 3.4.9. We have that $\mathcal{C}$ admits all K-indexed limits if and only if

$$\Delta : \mathcal{C} \to \mathrm{Fun}(K, \mathcal{C})$$

is a left adjoint. The right adjoint is given by $\lim_K(-)$.

Equalizers are limits along $\Delta^1 \amalg_{\partial\Delta^1} \Delta^1$, pullbacks are limits along $\Delta^1 \times \Delta^1 - (0,0)$, etc.

3.5 Localization



Definition 3.5.1. Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories. Let W be a collection of edges in $\mathcal{C}$, with no further assumption. Denote by $\mathrm{Fun}_W(\mathcal{C}, \mathcal{D})$ the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ spanned by functors $F\colon \mathcal{C} \to \mathcal{D}$ that carry edges of W into equivalences in $\mathcal{D}$. Formally, this is the pullback in sSet:

$$\begin{tikzcd} \mathrm{Fun}_W(\mathcal{C}, \mathcal{D}) \arrow[r] \arrow[d] & \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \arrow[d] \\ \mathrm{Fun}(W, \mathcal{D}^{\simeq}) \arrow[r] & \mathrm{Fun}(W, \mathcal{D}). \end{tikzcd}$$

A localization of $\mathcal{C}$ with respect to W is an ∞-category $\mathcal{C}[W^{-1}]$ together with a functor $\gamma\colon \mathcal{C} \to \mathcal{C}[W^{-1}]$ satisfying the following universal property. For any ∞-category $\mathcal{D}$, the functor γ induces an equivalence of ∞-categories:

$$\gamma^*\colon \mathrm{Fun}(\mathcal{C}[W^{-1}], \mathcal{D}) \xrightarrow{\simeq} \mathrm{Fun}_W(\mathcal{C}, \mathcal{D}).$$

The definition can be extended to any simplicial set $\mathcal{C}$, not necessarily an ∞-category.

The functor $hQCat \to \mathsf{Set}$ that is defined by:

$$\mathcal{D} \mapsto \pi_0(\mathrm{Fun}_W(\mathcal{C}, \mathcal{D})^{\simeq}) = \pi_0(\mathrm{Fun}(\mathcal{C}[W^{-1}], \mathcal{D})^{\simeq}) = \mathrm{Hom}_{hQCat}(\mathcal{C}[W^{-1}], \mathcal{D})$$

is corepresented by $\mathcal{C}[W^{-1}]$ and is thus unique up to isomorphism in $hQCat$, i.e. is unique up to equivalence of ∞-categories (if it exists).

Theorem 3.5.2. The localization $\gamma\colon \mathcal{C} \to \mathcal{C}[W^{-1}]$ always exists.

Before proving this, let us notice the following.

Example 3.5.3. Let $W \subseteq \Delta^1$ be the unique non-degenerate 1-simplex. Then $\Delta^1[W^{-1}] = \Delta^0$ and $\gamma\colon \Delta^1 \to \Delta^0$ is the localization. Indeed:

$$\mathcal{D} \simeq \mathrm{Fun}(\Delta^0, \mathcal{D}) \xrightarrow{\simeq} \mathrm{Fun}_W(\Delta^1, \mathcal{D}) = Eq(\mathcal{D})$$

where $Eq(\mathcal{D})$ are the equivalences in $\mathcal{D}$, defined on object X to id_X, is a trivial Kan fibration.

This observation can be extended to following.

Lemma 3.5.4. Let Q be a contractible Kan complex. Let $e\colon \Delta^1 \hookrightarrow Q$ be a monomorphism in sSet. Let $W \subseteq \Delta^1$ be the unique non-degenerate 1-simplex. Then there is an equivalence of ∞-categories:

$$\mathrm{Fun}(Q, \mathcal{D}) \xrightarrow{\simeq} \mathrm{Fun}_W(\Delta^1, \mathcal{D}) = Eq(\mathcal{D}).$$

Proof. Exercise. □

Proof of Theorem 3.5.2. Let $F\colon \mathcal{C} \to \mathcal{D}$ be in $\mathrm{Fun}_W(\mathcal{C}, \mathcal{D})$. For all $w \in W$, this defines $F(w)\colon \Delta^1 \to \mathcal{D}^{\simeq}$. Factor this morphism in the model category $\mathrm{sSet}_{\mathrm{Kan}}$:

$$\begin{tikzcd} \Delta^1 \arrow[rr, "F(w)"] \arrow[dr, hook, "\sim"'] & & \mathcal{D}^{\simeq} \\ & Q_w \arrow[ur, two heads, "q_w"'] & \end{tikzcd}$$

By construction, Q_w is a contractible Kan complex. We can consider the following pushout in sSet:

$$\begin{tikzcd} \coprod_{w \in W} \Delta^1 \arrow[r] \arrow[d, hook, "\sim"] & \mathcal{C} \arrow[d, "\gamma'"] \arrow[ddr, bend left, "G"] & \\ \coprod_{w \in W} Q_w \arrow[r] \arrow[drr, bend right] & \mathcal{C}' \arrow[dr, dashed, "\exists"] & \\ & & \mathcal{D} \end{tikzcd}$$

Given any ∞-category $\mathcal{D}$ fitting into the diagram above, notice $G(w) \in \mathcal{D}^{\simeq}$ by commutativity. Therefore the induced map by γ':

$$\mathrm{Fun}(\mathcal{C}', \mathcal{D}) \longrightarrow \mathrm{Fun}_W(\mathcal{C}, \mathcal{D})$$

is an equivalence of ∞-categories. Indeed in the diagram:

$$\begin{tikzcd}
\mathrm{Fun}(\mathcal{C}', \mathcal{D}) \arrow[r] \arrow[d] & \mathrm{Fun}_W(\mathcal{C}, \mathcal{D}) \arrow[r] \arrow[d] & \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \arrow[d] \\
\prod_{w \in W} \mathrm{Fun}(Q_w, \mathcal{D}) \arrow[r] & \prod_{w \in W} Eq(\mathcal{D}) \arrow[r] & \prod_{w \in W} \mathrm{Fun}(\Delta^1, \mathcal{D})
\end{tikzcd}$$

the right square is a pullback by definition of $\mathrm{Fun}_W(\mathcal{C}, \mathcal{D})$, the outer rectangle is a pullback since $\mathrm{Fun}(-, \mathcal{D})$ sends pushout to pullbacks. Therefore, the left square is a pullback. However, by the lemma, we know the left bottom map is a trivial Kan fibration, therefore the top left map is a trivial Kan fibration. Thus γ' defines an equivalence of ∞-categories as desired. We force now $\mathcal{C}'$ to be an ∞-category by performing a factorization in $\mathrm{sSet}_{\mathrm{Joyal}}$ on $\mathcal{C}' \to \mathcal{D}$ with a trivial cofibration and followed by fibration, which thus defines $\mathcal{C}[W^{-1}]$ with same property as $\mathcal{C}'$. $\square$

We can have a better descriptio of $\mathcal{C}[W^{-1}]$ when we have more assumption on W.

Definition 3.5.5. Let $\mathcal{C}$ be an ∞-category and W a collection of edges in $\mathcal{C}$. We say $Z \in \mathcal{C}$ is W-local if for all $w \colon X \to Y$ in W, we have a weak homotopy equivalence:

$$\mathrm{Hom}_{\mathcal{C}}(Y, Z) \xrightarrow{\simeq} \mathrm{Hom}_{\mathcal{C}}(X, Z).$$

We say W is localizing if:

- equivalences in $\mathcal{C}$ are in W;
- W satisfy 2-out-of-3;
- for all $Y \in \mathcal{C}$, there exists $w \colon Y \to Z$ in W such that Z is W-local.

Remark 3.5.6. If $w \colon X \to Y$ in W, X and Y are W-local, then w must be an equivalence.

Theorem 3.5.7. Suppose $\mathcal{C}$ is an ∞-category with W is localizing collection of edges, then $\mathcal{C}[W^{-1}]$ can be defined as the full subcategory spanned by the W-local objects and $\gamma \colon \mathcal{C} \to \mathcal{C}[W^{-1}]$ is a left adjoint:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, shift left, "\gamma"] \arrow[r, phantom, "\perp"] & \mathcal{C}[W^{-1}] \arrow[l, hook', shift left]
\end{tikzcd}$$

Proof. This follows by the previous remark and the universal property of $\mathcal{C}[W^{-1}]$. Give $X \in \mathcal{C}$, one can define informally $\gamma(X)$ by a choice of a map $w \colon X \to Y$ where Y is W-local, and given $X \to X'$ in $\mathcal{C}$, we can define a map $\gamma(X) \to \gamma(X')$:

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "\in W"'] & X' \arrow[d, "\in W"] \\
\gamma(X) \arrow[r] & \gamma(X').
\end{tikzcd}$$

and using the 2-out-of-3 property it is an equivalence whenever $X \to X'$ is in W. $\square$

Definition 3.5.8. Let $\mathcal{M}$ be a model category with W as class of weak equivalences. The Dwyer–Kan localization of $\mathcal{M}$ with W is the ∞-category $N(\mathcal{M})[W^{-1}]$ together with the localization $N(\mathcal{M}) \to N(\mathcal{M})[W^{-1}]$. This is sometimes referred as the underlying ∞-category of $\mathcal{M}$.

Remark 3.5.9. If $\mathcal{M}$ admits functorial fibrant and cofibrant replace, then:

$$N(\mathcal{M}_c)[W^{-1}] \simeq N(\mathcal{M}_f)[W^{-1}] \simeq N(\mathcal{M}_{cf})[W^{-1}] \simeq N(\mathcal{M})[W^{-1}].$$

How should one think of $N(\mathcal{M})[W^{-1}]$? Its objects are the objects in $\mathcal{M}$, but considered up to weak equivalence, the edges $X \to Y$ are elements in $\mathrm{Hom}_{\mathcal{M}}(X,Y)/\simeq$, the composition:

$$\begin{tikzcd} & Y \arrow[dr] & \\ X \arrow[ur] \arrow[rr, dashed] & & Z, \end{tikzcd}$$

is defined up to weak equivalence. In particular, every morphism can be considered to be a cofibration or a fibration. The homotopy relation in $N(\mathcal{M})[W^{-1}]$ is the same as defined in model categories. Notably, we obtain an equivalence of categories:

$$\mathrm{Ho}(N(\mathcal{M})[W^{-1}]) \simeq \mathrm{Ho}(\mathcal{M}).$$

Example 3.5.10. The ∞-category of spaces $\mathcal{S}$, i.e. the ∞-category of ∞-groupoids, is defined as the Dwyer–Kan localization $N(\mathrm{sSet})[W_{\mathrm{Kan}}^{-1}]$, and is denoted $\mathcal{S}$.

Example 3.5.11. The (large) ∞-category of ∞-categories is defined as the Dwyer–Kan localization $N(\mathrm{sSet})[W_{\mathrm{Joyal}}^{-1}]$ and is denoted Cat_∞.

Example 3.5.12. Let R be a commutative ring. Denote $\mathcal{D}(R)$ to the Dwyer–Kan localization of $N(\mathrm{Ch}_R)[W_{\mathrm{proj}}^{-1}]$.

Theorem 3.5.13 (HA 1.3.4.20)**.** If $\mathcal{M}$ is a combinatorial model category, then it is Quilllen equivalent to a simplicial model category $\widetilde{M}$ and $N(\mathcal{M})[W^{-1}]$ is equivalent to the homotopy coherent nerve of $\widetilde{M}_{cf}$.

Remark 3.5.14. If $\mathcal{M}$ is a simplicial model category, we can define $\mathcal{M}[W^{-1}]$ as a the hammock localization.

Definition 3.5.15. An ∞-category $\mathcal{C}$ is said to be compact if $\pi_0(\mathcal{C}^{\simeq})$ is compact as a set (i.e. small), and $\pi_i(\mathrm{Hom}_{\mathcal{C}}(X,Y))$ are compact as sets.

Definition 3.5.16. An object $X \in \mathcal{C}$ is compact if $\mathrm{Hom}_{\mathcal{C}}(X,-)\colon \mathcal{C} \to \mathcal{S}$ preserves filtered colimits.

Remark 3.5.17. An ∞-category $\mathcal{C}$ is compact if and only if it is compact as an object in Cat_∞.

Definition 3.5.18. An ∞-category $\mathcal{C}$ is said to be presentable if it has filtered colimits, and there exists an essentially small ∞-category $\mathcal{P} \subseteq \mathcal{C}$ comprised of compact objects which generates $\mathcal{C}$ under filtered colimits.

Proposition 3.5.19. An ∞-category $\mathcal{C}$ is presentable if and only if $\mathcal{C}$ is equivalent to $\mathrm{Fun}_W(\mathcal{P}^{\mathsf{op}}, \mathcal{S})$ for some small category $\mathcal{P}$ and some set of maps W in $\mathrm{Fun}(\mathcal{P}^{\mathsf{op}}, \mathcal{S})$.

Theorem 3.5.20. Let $\mathcal{C}$ be an ∞-category. Then $\mathcal{C}$ is presentable if and only if there exists a combinatorial model category $\mathcal{M}$ such that $\mathcal{C} \simeq N(\mathcal{M})[W^{-1}]$.

Presentable ∞-categories are combinatorial model categories.

Theorem 3.5.21. Let $\mathcal{M}$ be a combinatorial model category. Let J be a small category. Recall we can give $\mathrm{Fun}(J,\mathcal{M})$ the projective and injective model structures, both with weak equivalence defined levelwise. Evaluation $J \times \mathrm{Fun}(J,\mathcal{M}) \to \mathcal{M}$ lifts to a map $N(J) \times N(\mathrm{Fun}(J,\mathcal{M})) \to N(\mathcal{M})$ that induce an equivalence of ∞-categories:

$$N(\mathrm{Fun}(J,\mathcal{M}))[W_{\mathrm{Fun}}^{-1}] \xrightarrow{\simeq} \mathrm{Fun}(N(J), N(M)[W^{-1}]).$$

Proof. Universal property. $\square$

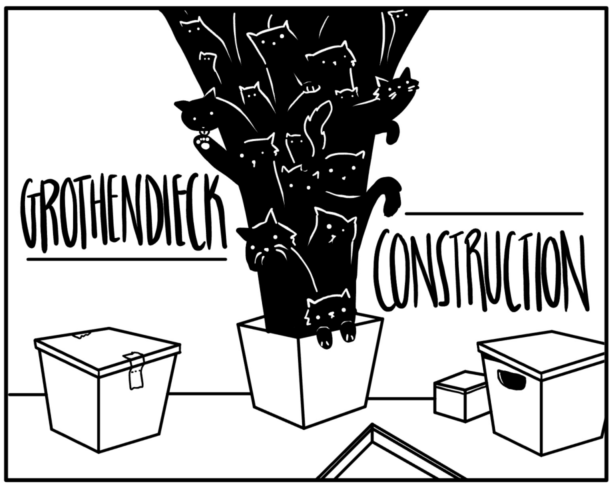
Theorem 3.5.22. Given a left Quillen functor $F\colon \mathcal{M}_1 \to \mathcal{M}_2$ between combinatorial model categories with weak equivalence classes denoted W_1 and W_2 respectively. Then the total left derived functor $\mathbb{L}F$ induces a functor on the Dwyer–Kan localizations:

$$\mathbb{L}F\colon N(\mathcal{M}_1)[W_1^{-1}] \longrightarrow N(\mathcal{M}_2)[W_2^{-1}]$$

that is a left adjoint.

Corollary 3.5.23. Let $\mathcal{M}$ be a combinatorial model category. Then colimits in $N(\mathcal{M})[W^{-1}]$ correspond precisely to homotopy colimits in $\mathcal{M}$. Similarly, limits in $N(\mathcal{M})[W^{-1}]$ correspond precisely to homotopy limits in $\mathcal{M}$.

3.6 Straightening/unstraightening— Higher categorical Grothendieck construction



Motivation: Let X be a space, and let $\mathrm{Cov}(X)$ denote the 1-category of covering spaces of X, so that in particular the fibers f^{-1} of $f : E \to X$ are discrete sets. This defines a map in Top from

$$X \to \mathsf{Set}^{\cong},$$

to sets with the discrete topology. Another way to think about this is as a functor

$$\begin{aligned} \mathrm{St} : \mathrm{Cov}(X) &\to \mathrm{Fun}(\Pi_1(X), \mathsf{Set}) \\ (E \xrightarrow{p} X) &\mapsto \left[x \mapsto f^{-1}(x)\right]. \end{aligned}$$

A path from x to y (a morphism in $\Pi_1(X)$) induces a set map $f^{-1}(x) \to f^{-1}(y)$.

This is an equivalence of categories! This is called the *fundamental theorem of covering spaces.*

This is a first instance of *straightening.*

If we view X as an ∞-groupoid, then $\Pi_1(X) = \mathrm{Ho}(X)$ is its homotopy category, and we have that

$$\mathrm{Fun}(\Pi_1(X), \mathsf{Set}) \cong \mathrm{Fun}(X, N(\mathsf{Set})),$$

since nerve is right adjoint to the homotopy category.

We can denote by $\mathrm{Cov}_X \subseteq \mathcal{S}/X$ to be the full subcategory of the infinity category of spaces over X spanned by covering spaces. Then we want to show that

$$\mathrm{Cov}_X \simeq \mathrm{Fun}(X, N(\mathsf{Set})).$$

We have an unstraightening functor

$$\mathrm{Unst} : \mathrm{Fun}(X, N(\mathsf{Set})) \to \mathrm{Cov}_X,$$

given by sending some $F : X \to N(\mathsf{Set})$ to the pullback[5]

$$\begin{tikzcd} E \arrow[r] \arrow[d] & N(\mathsf{Set}_*)^{\simeq} \arrow[d] \\ X \arrow[r, "F"'] & N(\mathsf{Set})^{\simeq} \end{tikzcd}$$

More generally, if we don't require the fibers to be discrete, then we can take $f : E \to X$ to be any continuous map. Then we get a functor[6]

$$\begin{aligned} \mathrm{St} : \mathcal{S}/X &\to \mathrm{Fun}(X, \mathcal{S}) \\ (E \xrightarrow{f} X) &\mapsto \left[x \mapsto f^{-1}(x)\right]. \end{aligned}$$

Unstraightening is of the form

$$\begin{aligned} \mathrm{Unst} : \mathrm{Fun}(X, \mathcal{S}) &\to \mathcal{S}/X \\ F &\mapsto \mathrm{hocolim}_X F = \cup_{x \in X} F^{-1}(x)/\sim . \end{aligned}$$

Let X be connected and suppose $X \simeq BG$. Then we define *G-modules in spaces* to be

$$\mathrm{Mod}_G(\mathcal{S}) := \mathrm{Fun}(BG, \mathcal{S}) \xrightarrow{\sim} \mathcal{S}/BG.$$

If we take some $M : BG \to \mathcal{S}$, and we post-compose with sections $\mathcal{S}/BG \to \mathcal{S}$, then M maps to M^{hG}.

More generally, given $F : X \to \mathcal{S}$, the limit $\lim_X \mathcal{S}$ is given by

$$\mathrm{Fun}(X, \mathcal{S}) \xrightarrow{\mathrm{Unst}} \mathcal{S}/X \xrightarrow{\text{sections}} \mathcal{S}.$$

Goal: Generalize this approach where X is replaced by an ∞-category $\mathcal{C}$ and $\mathcal{S}$ is replaced by Cat_∞. That is, we want to relate $\mathrm{Fun}(\mathcal{C}, \mathsf{Cat}_\infty)$ with some subcategory of $\mathsf{Cat}_\infty/\mathcal{C}$.

If $f : \mathcal{E} \to \mathcal{C}$, what requirement do we need to make sense of an associated functor

$$\begin{aligned} F : \mathcal{C} &\to \mathsf{Cat}_\infty \\ X &\mapsto f^{-1}(X). \end{aligned}$$

That is, how can we coherently choose our fibers.

Given $X \in \mathcal{C}$, we could take a pullback in Cat_∞:

$$\begin{tikzcd} f^{-1}(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{E} \arrow[d] \\ \Delta^0 \arrow[r, "X"'] & \mathcal{C}. \end{tikzcd}$$

If we choose $\mathrm{sSet}_{\mathrm{Joyal}}$ as our model, we would need $\mathcal{E} \to \mathcal{C}$ to be an inner fibration (RLP wrto inner horns) to get the pullback $f^{-1}(X)$ to be a quasi-category. If we instead say "pullback in quasi-categories," this requirement goes away.

Given $f : \mathcal{E} \to \mathcal{C}$ and $X \to Y$ in $\mathcal{C}$, how can we define $f^{-1}(X) \to f^{-1}(Y)$ in Cat_∞?

Need: If $\phi : X \to Y$ in $\mathcal{C}$ and $E_X \in \mathcal{E}$ such that $f(E_X) = X$, then there exists some $E_Y \in \mathcal{E}$ and $\phi_! : E_X \to E_Y$ in $\mathcal{E}$ so that $f(\phi_!) = \phi$, and that is universal in the following sense: for all $Z \in \mathcal{C}$ and for all $\psi : X \to Z$ in $\mathcal{C}$ for all $\bar{\psi} : E_X \to E_Z$ in $\mathcal{E}$ where $f(\bar{\psi}) = \psi$, if there exists $\gamma : Y \to Z$ then there exists a unique map $\bar{\gamma} : E_Y \to E_Z$ in $\mathcal{E}$ so that $f(\bar{\gamma}) = \gamma$ and $\bar{\gamma} \circ \phi_! = \bar{\psi}$.

We say that $\phi_! : E_X \to E_Y$ is a *cocartesian lift* of ϕ.

[5]Note that $N(\mathsf{Set}^{\simeq}) = N(\mathsf{Set})^{\simeq}$.

[6]By $\mathrm{Fun}(X, \mathcal{S})$ we might mean $\mathrm{Fun}(\mathrm{Sing}(X), N_\Delta(\mathrm{Kan}))$.

Definition 3.6.1. We say that $f : \mathcal{E} \to \mathcal{C}$ is a *cocartesian fibration* if for all $E_X \in \mathcal{E}$, for all $\phi : X \to Y$ with $f(E_X) = X$, there exists a cocartesian lift of ϕ.

Two cocartesian lifts over the same map are equivalent.

Given $f : \mathcal{E} \to \mathcal{C}$, $X \in \mathcal{C}$, $\phi : X \to Y$ in $\mathcal{C}$, we say $\phi_! : E_X \to E_Y$ is a cocartesian lift if the following is a pullback diagram in spaces:

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{E}}(E_Y, E_Z) \arrow[r, "(\phi_!)^*"] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Hom}_{\mathcal{C}}(E_X, E_Z) \arrow[d, "f"] \\ \mathrm{Hom}_{\mathcal{C}}(Y, Z) \arrow[r, "\phi^*"'] & \mathrm{Hom}_{\mathcal{C}}(X, Z), \end{tikzcd}$$

for any $Z \in \mathcal{C}$. In particular, taking maps from Δ^0 to the top right and bottom left picks out $\bar{\psi}$ and γ, respectively, so that $\gamma \circ \phi = \psi$, and the universal property of the pullback says that there exists $\bar{\gamma} : E_Y \to E_Z$ so that $\bar{\gamma}\phi_! = \bar{\psi}$ and $f(\bar{\gamma}) = \gamma$.

Definition 3.6.2. We define $\mathrm{coCart}(\mathcal{C}) \subseteq \mathsf{Cat}_\infty/\mathcal{C}$ to be the subcategory of cocartesian fibrations $\mathcal{E} \to \mathcal{C}$, with morphisms

$$\begin{tikzcd} \mathcal{E} \arrow[rr, "G"] \arrow[dr, "f"'] & & \mathcal{E}' \arrow[dl, "f'"] \\ & \mathcal{C}, & \end{tikzcd}$$

so that G sends f-cocartesian lifts to f'-cocartesian lifts.

In this case, straightening defines a functor

$$\mathrm{St} : \mathrm{coCart}(\mathcal{C}) \to \mathrm{Fun}(\mathcal{C}, \mathsf{Cat}_\infty),$$

sending $f : \mathcal{E} \to \mathcal{C}$ to the functor

$$\begin{aligned} \mathcal{C} &\to \mathsf{Cat}_\infty \\ X &\mapsto f^{-1}(X) \\ (X \xrightarrow{\phi} Y) &\mapsto \left[f^{-1}(X) \xrightarrow{\phi_!} f^{-1}(Y) \right]. \end{aligned}$$

Example 3.6.3. Let $f : X \to Y$ in $\mathcal{S}$. All lifts are cocartesian lifts. We say that a *left fibration* is a cocartesian fibration where every lift is cocartesian.

Example 3.6.4. Suppose $\mathcal{C}$ is an ordinary category. Then we can define a new category whose objects are $f : X \to Y$ in $\mathcal{C}$, and whose morphisms are

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[d, "v"] \\ X' \arrow[u, "u"] \arrow[r, "f'"'] & Y'. \end{tikzcd}$$

This defines what we call the *twisted arrow category* $\mathrm{Tw}(\mathcal{C})$. There is a natural functor

$$\begin{aligned} \mathrm{Tw}(\mathcal{C}) &\xrightarrow{\mathrm{Ev}} \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \\ (X \xrightarrow{f} Y) &\mapsto (X, Y). \end{aligned}$$

This is a left fibration, by composition. Straightening this, we get

$$\begin{aligned} \mathrm{St}(\mathrm{Ev}) : \mathcal{C}^{\mathsf{op}} \times \mathcal{C} &\to \mathsf{Set} \\ (X, Y) &\mapsto \mathrm{Ev}^{-1}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(X, Y). \end{aligned}$$

Example 3.6.5. If $\mathcal{C}$ is an ∞-category, we can define a twisted arrow category in a similar way

$$\begin{aligned} \mathrm{Tw}(\mathcal{C}) : \Delta^{\mathsf{op}} &\to \mathsf{Set} \\ [n] &\mapsto \mathrm{Hom}_{\mathrm{sSet}}(\Delta^{2n+1}, \mathcal{C}), \end{aligned}$$

where the n-simplices of $\mathrm{Tw}(\mathcal{C})$ should be thought of as

$$\begin{tikzcd} X_0 \arrow[d] & X_1 \arrow[l] \arrow[d] & X_2 \arrow[l] \arrow[d] & \cdots \arrow[l] & X_n \arrow[l] \arrow[d] \\ Y_0 \arrow[r] & Y_1 \arrow[r] & Y_2 \arrow[r] & \cdots \arrow[r] & Y_n. \end{tikzcd}$$

We can define

$$\begin{aligned} \ell &: \mathrm{Tw}(\mathcal{C}) \to \mathcal{C}^{\mathsf{op}} \\ r &: \mathrm{Tw}(\mathcal{C}) \to \mathcal{C}, \end{aligned}$$

by precomposition with $\Delta^n \hookrightarrow \Delta^{2n+1}$. These assemble to give

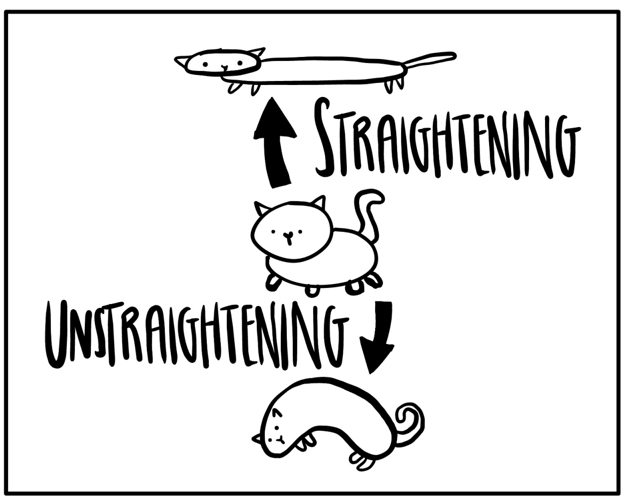
$$\mathrm{Tw}(\mathcal{C}) \xrightarrow{\mathrm{Ev}} \mathcal{C}^{\mathsf{op}} \times \mathcal{C},$$

and we have $\mathrm{Hom}_{\mathcal{C}}(X, Y) = \mathrm{Ev}^{-1}(X, Y) \in \mathcal{S}$. This evaluation map is a left fibration, left fibrations are preserved under pullback, and left fibrations over Δ^0 are Kan complexes. Therefore $\mathrm{Ev}^{-1}(X)$ is a space.

Example 3.6.6. Let $X \in \mathcal{C}$. Then we can take

$$\begin{tikzcd} \ell^{-1}(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Tw}(\mathcal{C}) \arrow[d, "\ell"] \\ \Delta^0 \arrow[r, "X"'] & \mathcal{C}^{\mathsf{op}}. \end{tikzcd}$$

We define $\mathcal{C}_{X/} := \ell^{-1}(X)$, and $r^{-1}(Y) := \mathcal{C}_{/Y}$.



Theorem 3.6.7. *(Straightening/unstraightening)* If $\mathcal{C}$ is an ∞-category, we can define its *unstraightening* as

$$\begin{aligned} \mathrm{Unst} : \mathrm{Fun}(\mathcal{C}, \mathsf{Cat}_\infty) &\to \mathrm{coCart}(\mathcal{C}) \\ F &\mapsto \mathrm{colim}\left(\mathrm{Tw}(\mathcal{C}) \xrightarrow{\mathrm{Ev}} \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \xrightarrow{\mathcal{C}_{/\cdot} \times F} \mathsf{Cat}_\infty\right). \end{aligned}$$

That composite sends

$$\mathrm{Tw}(\mathcal{C}) \xrightarrow{\mathrm{Ev}} \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \xrightarrow{\mathcal{C}_{/\cdot} \times F} \mathsf{Cat}_\infty$$
$$(X \xrightarrow{f} Y) \mapsto \mathcal{C}_{X/} \times F(Y).$$

This forms an equivalence with St.

There is an equivalence

$$\mathrm{St} : \mathrm{LFib}(\mathcal{C}) \leftrightarrows \mathrm{Fun}(\mathcal{C}, \mathcal{S}) : \mathrm{Unst}.$$

If $\mathcal{C} = X \in \mathcal{S}$, then $\mathrm{coCart}(X) = \mathsf{Cat}_\infty/X$.
If $\mathcal{C} = N(\mathcal{D})$, this recovers the usual Grothendieck construction.
If $F : \mathcal{C} \to \mathsf{Cat}_\infty$, then

$$\operatorname{colim} F = \mathrm{Unst}(\mathcal{C})[\text{cocart. edges}^{-1}]$$

Chapter 4

Higher algebraic structures

4.1 Unstraightening multiplications

Recall $\mathcal{S} \simeq N(\text{sSet})[W_{\text{Kan}}^{-1}]$ the ∞-category of spaces. When we say $X \to Y$ is a map in $\mathcal{S}$ we mean that $X \to Y$ is a map in Ho(sSet) *not* that $X \to Y$ is any map in sSet.

Example 4.1.1. If we have $X \to Y$ in $\mathcal{S}$, then $X \to Y$ is a left fibration. If X and Y are in Kan and $X \to Y$ this *does not imply* that $X \to Y$ must be a left fibration. What is true is that if $X \to Y$ is a Kan fibration, then $X \to Y$ is a left fibration.

We have $\mathsf{Cat}_\infty \simeq N(\text{sSet})[W_{\text{Joyal}}^{-1}]$, so $f : \mathcal{C} \to \mathcal{D}$ in Cat_∞ means

$$\begin{tikzcd} f^{-1}(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{C} \arrow[d] \\ \Delta^0 \arrow[r, "X"'] & \mathcal{D}. \end{tikzcd}$$

So we always want it to be a fibration.

That is, a map $f : \mathcal{C} \to \mathcal{D}$ in Cat_∞ is not the same as $\mathcal{C} \to \mathcal{D}$ of quasi-categories in sSet.

In Cat_∞, $\mathcal{C} \to \mathcal{D}$ is a cocartesian fibration if there exists a cocartesian lift on any fiber.

If $\mathcal{C}, \mathcal{D}$ are quasi-categories in $\text{sSet}_{\text{Joyal}}$, then $f : \mathcal{C} \to \mathcal{D}$ is a cocartesian fibration if f is an *inner fibration* (RLP inner horns) AND there is a cocartesian lift of any fiber. The inner fibration condition guarantees that the fibers are also infinity categories.

Straightening definition last time was wrong. Last time, we had

$$\begin{aligned} \text{Unst} : \text{Fun}(\mathcal{C}, \mathsf{Cat}_\infty) &\xrightarrow{\sim} \text{coCart}(\mathcal{C}) \\ F &\mapsto \left(\mathcal{E} \xrightarrow{\text{Unst}(F)} \mathcal{C}\right). \end{aligned}$$

is an equivalence of categories, where

$$\mathcal{E} = \text{colim}\left(\text{Tw}(\mathcal{C})^{\mathsf{op}} \to \mathcal{C} \times \mathcal{C}^{\mathsf{op}} \xrightarrow{F \times \mathcal{C}_{\bullet/}} \mathsf{Cat}_\infty\right).$$

Example 4.1.2. Take $\mathcal{C} = *$. Then $\text{Fun}(*, \mathsf{Cat}_\infty) = \mathsf{Cat}_\infty$. We have that $\text{coCart}(*) = \mathsf{Cat}_\infty$, and that $\text{Tw}(*) = *^{\mathsf{op}} = *$. The composite sends

$$\begin{aligned} \text{Tw}(*)^{\mathsf{op}} \to * \times *^{\mathsf{op}} &\to \mathsf{Cat}_\infty \\ * \mapsto (*, *) &\mapsto *A \times * = A. \end{aligned}$$

Example 4.1.3. Take $\mathcal{C} = 1 = 0 \to 1$. A functor $F : 1 \to \mathsf{Cat}_\infty$ is exactly a functor $F : \mathcal{A} \to \mathcal{D}$ in Cat_∞. We see that $\mathrm{Tw}(1)$ has three objects, being $0 = 0$, $0 \to 1$ and $1 = 1$. The identity ones both map to $0 \to 1$ so it is a span-op category. When we op $\mathrm{Tw}(1)^{\mathsf{op}}$ we get the span category, so a colimit becomes a pushout. We see that $1_{0/} = 1$ and $1_{1/} = *$. Then

$$\mathcal{E} = \operatorname{colim}\left(\begin{tikzcd} \mathcal{A} \times 1_{1/} \arrow[r, "\mathrm{id}\times(0\to1)"] \arrow[d, "F\times\mathrm{id}"'] & \mathcal{A} \times 1_{0/} \\ \mathcal{B} \times 1_{1/} & \end{tikzcd}\right)$$

$$= \operatorname{colim}\left(\begin{tikzcd} \mathcal{A} \arrow[r, "\mathrm{id}\times 1"] \arrow[d] & \mathcal{A} \times 1 \\ \mathcal{B} & \end{tikzcd}\right)$$

Then $\mathcal{E}$ is a cocartesian fibration over 1, whose fiber over 0 is $\mathcal{A}$, whose fiber over 1 is $\mathcal{B}$, and with maps $F(A) \to B$ over $0 \to 1$.

Goal: Redefine a symmetric monoidal category $(\mathcal{C}, \otimes, I)$ as a cocartesian fibration $\mathcal{C}^\otimes \to \mathrm{Fin}_*$ as certain "pseudo"functors $\mathrm{Fin}_* \to \mathsf{Cat}$. We could take $\mathrm{Fin}_* \to \mathsf{Cat}$ sending $\langle n\rangle$ to $\mathcal{C}^{\times n}$.

Q: Given a psuedofunctor $F : \mathrm{Fin}_* \to \mathsf{Cat}$, when is it defining a symmetric monoidal category?

We would need $F(\langle n\rangle) \cong F(\langle 1\rangle)^{\times n}$ with Segal's condition $F(\langle 0\rangle) = 0$.

Theorem 4.1.4. Symmetric monoidal categories are pseudofunctors $\mathrm{Fin}_* \to \mathsf{Cat}$ with the Segal condition.

4.2 Algebras

Last time we defined a symmetric monoidal infinity category to be a cocartesian fibration over Fin_* with a Segal condition. Here $\mathcal{C} = f^{-1}(\langle 1\rangle)$. We got this by straightening $N(\mathrm{Fin}_*) \to \mathsf{Cat}_\infty$, with $\langle n\rangle \mapsto \mathcal{C}^{\otimes n}$.

Suppose we had a natural transformation η between functors

$$\mathcal{C}, \mathcal{D} : N(\mathrm{Fin}_*) \to \mathsf{Cat}_\infty.$$

This corresponds to a map $\mathcal{C}^\otimes \to \mathcal{D}^\otimes$ over Fin_* sending p-cocartesian lifts to q-cocartesian lifts:

$$\begin{tikzcd} \mathcal{C}^\otimes \arrow[rr] \arrow[dr, "p"'] & & \mathcal{D}^\otimes \arrow[dl, "q"] \\ & N(\mathrm{Fin}_*). & \end{tikzcd}$$

Think about this as $F(X) \otimes F(Y) \xrightarrow{\sim} F(X \otimes Y)$.

Now suppose we have $F^\otimes : \mathcal{C}^\otimes \to \mathcal{D}^\otimes$ between symmetric monoidal ∞-categories. Then we know the fiber over $\langle 1\rangle$ must be sent to the fiber over $\langle 1\rangle$. Then we get $F^\otimes_{\langle n\rangle} : \mathcal{C}^\otimes_{\langle n\rangle} \to \mathcal{D}^\otimes_{\langle n\rangle}$ for all n.

Denote $F = F^\otimes_{\langle 1\rangle}$. Then $F^\otimes_{\langle n\rangle} \simeq F^{\times n}$.

Let $\rho^i_! : \langle n\rangle \to \langle 1\rangle$ send everything to 0 except i to 1.

$$\begin{tikzcd} \mathcal{C}^\otimes_{\langle 2\rangle} \arrow[r, "F^\otimes_{\langle 2\rangle}"] \arrow[d, "(\rho^1_!, \rho^2_!)"'] & \mathcal{D}^\otimes_{\langle 2\rangle} \arrow[d, "(\rho^1_!, \rho^2_!)"] \\ \mathcal{C} \times \mathcal{C} \arrow[r, "F\times F"'] & \mathcal{D} \times \mathcal{D}. \end{tikzcd}$$

$F(\rho^1_!) \simeq \rho^1_!$ and $F(\rho^2_!) \simeq \rho^2_!$. For all i we need that $F(\rho^i_!)$ is a q-cocartesian lift of ρ^i. This means that for all n, $F^{\otimes}_{\langle n \rangle}(X_1, \dots, X_n) \simeq (F(X_1), \dots, F(X_n))$.

Definition 4.2.1. A map $\alpha : \langle n \rangle \to \langle k \rangle$ in Fin_* is *inert* if $\alpha^{-1}(i)$ is precisely a singleton for $1 \leq i \leq n$.

Fact 4.2.2. Inert morphisms are generated by ρ^i and τ (here τ is the swap of 1 and 2 on $\langle 2 \rangle$).

Let $F^{\otimes} : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ that sends p-cocartesian lifts of inert maps to q-cocartesian lifts. We claim this already gives a lax monoidal structure. Consider $m : \langle 2 \rangle \to \langle 1 \rangle$ the multiplication, and consider $(X, Y) \in \mathcal{C}^{\times 2}$. There is a map $m_! : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ sending $(X, Y) \mapsto X \otimes Y$.

$$\begin{tikzcd}
 & F(X) \otimes F(Y) \arrow[dr, dashed] & \\
(F(X), F(Y)) \arrow[ur, "m_!"] \arrow[rr, "F(m_!)"'] & & F(X \otimes Y).
\end{tikzcd}$$

Note we're not saying that $F(m_!)$ is a cocartesian lift, we're saying that $m_!$ is. If $F(m_!)$ was a cocartesian lift, then this would give $F(X) \otimes F(Y) \to F(X \otimes Y)$ is an equivalence.

Exercise 4.2.3. Show that $\iota : \langle 0 \rangle \to \langle 1 \rangle$ induces $I_{\mathcal{D}} \to F(I_{\mathcal{C}})$.

Definition 4.2.4. For $\mathcal{C}^{\otimes}$ and $\mathcal{D}^{\otimes}$ symmetric monoidal ∞-categories, a *lax symmetric monoidal functor* $F^{\otimes} : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ is a functor that sends lifts of p-cocartesian inert maps in Fin_* to q-cocartesian lifts.

Definition 4.2.5. We say $F^{\otimes}$ is strong symmetric monoidal if it sends *all* p-cocartesian lifts to q-cocartesian lifts.

We can define

$$\begin{tikzcd}
\mathrm{Fun}_{N(\mathrm{Fin}_*)}(\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Fun}(\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes}) \arrow[d, "q^*"] \\
\Delta^0 \arrow[r, "p"'] & \mathrm{Fun}(\mathcal{C}^{\otimes}, \mathrm{Fin}_*).
\end{tikzcd}$$

Define $\mathrm{Fun}^{\otimes,\mathrm{lax}}(\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes})$ to be the full subcategory of lax monoidal functors, and just $\mathrm{Fun}^{\otimes}(\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes})$ the full subcategory of strong monoidal functors.

Example 4.2.6. Commutative algebras. We have that Δ^0 is a symmetric monoidal ∞-category with trivial structure, then we have

$$N(\mathrm{Fin}_*) \to \mathsf{Cat}_{\infty}$$

sending everything to Δ^0. The associated cocartesian fibration is $N(\mathrm{Fin}_*) \to N(\mathrm{Fin}_*)$.

We define $\mathrm{Alg}_{\infty}(\mathcal{C})$ to be $\mathrm{Fun}^{\otimes,\mathrm{lax}}(N(\mathrm{Fin}_*), \mathcal{C})$. That is,

$$\begin{tikzcd}
N(\mathrm{Fin}_*) \arrow[rr, "A^{\otimes}"] \arrow[dr, equal] & & \mathcal{C}^{\otimes} \arrow[dl, "p"] \\
 & N(\mathrm{Fin}_*) &
\end{tikzcd}$$

That is, $A^{\otimes}$ is a section of p that sends inert maps in Fin_* to p-cocartesian lifts. We have that $A^{\otimes}(\langle 1 \rangle) \in \mathcal{C}^{\otimes}_{\langle 1 \rangle} = \mathcal{C}$, and $A \otimes A \to A$. We have that $A^{\otimes}(\langle 0 \rangle) = I$.

Q: Can we localize a symmetric monoidal category in such a way that it preserves the symmetric monoidal structure?

Definition 4.2.7. (HA 4.1.7.4) Given $\mathcal{C}^{\otimes}$ a symmetric monoidal ∞-category, let $W \subseteq \mathcal{C}$ a collection of edges. Assume W is closed under $\otimes$ (meaning that if $Y \to Y'$ is in W, and X is arbitrary, then $X \otimes Y \to X \otimes Y'$ and $Y \otimes X \to Y' \otimes X$ are in W as well). The *symmetric monoidal localization* of $\mathcal{C}^{\otimes}$ with W is a symmetric monoidal ∞-category $\mathcal{C}[W^{-1}]^{\otimes}$ together with a strong symmetric monoidal functor

$$\ell : \mathcal{C}^{\otimes} \to \mathcal{C}[W^{-1}]^{\otimes}$$

with the following universal property: for any symmetric monoidal ∞-category $\mathcal{D}^{\otimes}$, we get an equivalence of ∞-categories:

$$\mathrm{Fun}^{\otimes}(\mathcal{C}[W^{-1}]^{\otimes}, \mathcal{D}^{\otimes}) \xrightarrow{\sim} \mathrm{Fun}^{\otimes}_{W}(\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes}),$$

where $\mathrm{Fun}_W(-)$ means sending W to equivalences.

This always exists. We have that $\mathcal{C}[W^{-1}]_{\langle 1 \rangle} \simeq \mathcal{C}[W^{-1}]$. In terms of cocartesian fibrations it is maybe(?) some kind of Kan extension

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d] & \mathrm{Fin}_* \\ \mathcal{C}[W^{-1}] \arrow[ur, dashed, "?"] & \end{tikzcd}$$

Definition 4.2.8. Let $(M, \otimes, I)$ be a symm mon model category (with functorial cofibrant replacement). Suppose I is cofibrant. Then the Dwyer-Kan localization $N(M)[W^{-1}]$ can be given a symmetric monoidal ∞-structure as follows:

- Take the cofibrant objects M_c
- Take the category of operators $M_c^{\otimes}$ as an ordinary category (objects are pairs $\langle n \rangle, c_1, \ldots, c_n$) and morphisms are $\otimes_i c_i \to c'_j$ over Fin_*
- $N(M_c^{\otimes})$ is a symmetric monoidal ∞-category, with class W of edges in $N(M_c)$
- Recall that $X \otimes - : M_c \to M_c$ preserves weak equivalences between cofibrant objects, under the hypothesis that X is cofibrant.
- Thus $N(M_c^{\otimes}) \to N(M_c)[W^{-1}]^{\otimes}$ is called the *symmetric monoidal Dwyer-Kan localization.*

This gives a sym mon structure on the ∞-category $N(M)[W^{-1}] \simeq N(M_c)[W^{-1}]$.

This shows that the derived tensor product $\otimes$ of a monoidal model category M endows $N(M)[W^{-1}]$ with a monoidal structure.

Example 4.2.9. Spaces $\mathcal{S}$ have a symmetric monoidal ∞-category structure, since we can view them as $N(\mathrm{sSet})[W_{\mathrm{Kan}}^{-1}]$ with the cartesian product. Here $\mathrm{Alg}_{E_\infty}(\mathcal{S})$ are equivalent to E_∞-algebras in spaces.

Example 4.2.10. We have that $\mathsf{Cat}_\infty \simeq N(\mathrm{sSet})[W_{\mathrm{Joyal}}^{-1}]$ with the cartesian product. Then $\mathrm{Alg}_{E_\infty}(\mathsf{Cat}_\infty)$ are symmetric monoidal ∞-categories. This is exactly because $\mathrm{Alg}_{E_\infty}(\mathsf{Cat}_\infty) = \mathrm{Fun}^{\otimes,\mathrm{lax}}(N(\mathrm{Fin}_*), \mathsf{Cat}_\infty)$ which guarantees the Segal condition.

Example 4.2.11. If R is a commutative ring, then $D(R) \simeq \mathrm{Ch}_R[W_{\mathrm{proj}}^{-1}]$ is a symmetric monoidal ∞-category. The injective model structure does not give you a monoidal model category.

We also have the connective case with two models

$$D^{\geq 0}(R) \simeq N(s\mathrm{Mod}_R)[W^{-1}] \simeq N(\mathrm{Ch}_R^{\geq 0})[W^{-1}].$$

Every symmetric monoidal ∞-category $\mathcal{C}^{\otimes}$ which is presentable and for which $\otimes$ preserves colimits is the symmetric monoidal DK localization of a combinatorial monoidal model category (Lurie-Sagave).

4.3 Stable ∞-categories

Universal property for $\mathcal{S}$ (spaces). Given $K \in \text{sSet}$, there is a Yoneda embedding

$$K \hookrightarrow \text{Fun}(K^{\mathsf{op}}, \mathcal{S}) =: \mathcal{P}(K),$$

which is the adjoint of "internal hom"[1]

$$K^{\mathsf{op}} \times K \to \mathcal{S}.$$

Given $\mathcal{C}$ an ∞-category, we can call $\mathcal{P}(\mathcal{C})$ the *universal cocompletion* of $\mathcal{C}$. That is, for all $\mathcal{D}$ cocomplete, there is an equivalence

$$\text{Fun}^L(\mathcal{P}(\mathcal{C}), \mathcal{D}) \xrightarrow{\sim} \text{Fun}(\mathcal{C}, \mathcal{D}),$$

where L denotes colimit-preserving functors.[2]

If we choose $\mathcal{C} = \Delta^0$, we get

$$\text{Fun}^L(\mathcal{S}, \mathcal{D}) = \text{Fun}(\Delta^0, \mathcal{D}) = \mathcal{D}.$$

Hence we can think of $\mathcal{S}$ as the "free cocompletion of Δ^0." Just as a set can be viewed as a union of its points, we can think of any cocomplete ∞-category as gluing its paths together.

Definition 4.3.1. An ∞-category is *pointed* if it has an object with is both initial and terminal. That is, some $0 \in \mathcal{C}$ so that

$$\text{Hom}_{\mathcal{C}}(0, X) \simeq * \simeq \text{Hom}_{\mathcal{C}}(X, 0)$$

for any $X \in \mathcal{C}$.

Example 4.3.2. If $\mathcal{C}$ is an ∞-category and $* \in \mathcal{C}$ is a terminal object, we can define

$$\mathcal{C}_* := \mathcal{C}_{*/}.$$

This will be pointed and we will have an adjunction

$$(-)_+ : \mathcal{C} \leftrightarrows \mathcal{C}_*.$$

For example, we have

$$\mathcal{S} \leftrightarrows \mathcal{S}_* = N(\text{sSet}_*)[W_{\text{Kan}}^{-1}].$$

If $\mathcal{C}$ is a pointed presentable stable ∞-category, then

$$\text{Fun}^L(\mathcal{S}_*, \mathcal{C}) \simeq \mathcal{C}.$$

Here $\mathcal{S}_*$ is the free presentable pointed ∞-category generated by $*_+ = S^0$.

Now we introduce stable ∞-categories, which behave like $D(R) \simeq N(\text{Ch}_R)[W_{\text{qiso}}^{-1}]$.

Definition 4.3.3. Let $\mathcal{C}$ be a pointed ∞-category. A *triangle* in $\mathcal{C}$ is a square of the form

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d, "g"] \\ 0 \arrow[r] & Z. \end{tikzcd}$$

This is specified by a functor $N(\Delta^1 \times \Delta^1) \to \mathcal{C}$ sending the bottom corner to 0.

[1] K isn't necessarily an ∞-category, so it doesn't make sense to have internal hom, but this is the straightening of $\text{Tw}(K) \to K^{\mathsf{op}} \times K$ which is always well-defined.

[2] For presentable ∞-categories, being a left adjoint is equivalent to preserving colimits, hence the superscript "L"

We say a triangle is *exact* if it is a pullback, and *coexact* if it is a pushout.

Example 4.3.4. If $f : E \to X$ in $\mathcal{S}_*$, then an exact triangle looks like

$$\begin{tikzcd} f^{-1}(x) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "f"] \\ * \arrow[r] & X. \end{tikzcd}$$

Example 4.3.5. We have loops and suspension in $\mathcal{S}_*$ given by the (homotopy) pullback and pushout squares

$$\begin{tikzcd} \Omega X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & * \arrow[d] \\ * \arrow[r] & X \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] \arrow[ur, phantom, "\ulcorner", very near start] & \Sigma X \end{tikzcd}$$

Our goal is to define $\Sigma : \mathcal{C} \to \mathcal{C}$ and $\Omega : \mathcal{C} \to \mathcal{C}$ for a general pointed ∞-category.

Definition 4.3.6. For $\mathcal{C}$ finitely bicomplete, we define $\mathcal{C}^{\Sigma} \subseteq \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{C})$ to be the full subcategory spanned by diagrams of the form

$$\begin{tikzcd} X \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & \Sigma X \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

Note that maps between such diagrams are the same as maps $X \to Y$. Thus there is an equivalence

$$\mathcal{C}^{\Sigma} \xrightarrow{\sim} \mathcal{C},$$

and similarly $\mathcal{C}^{\Omega} \xrightarrow{\sim} \mathcal{C}$.

We have

$$\begin{tikzcd} \Gamma \arrow[r] \arrow[d, "\sim"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}(\mathcal{C}, \mathcal{C}^{\Sigma}) \arrow[d, "\simeq"] \\ * \arrow[r, "\mathrm{id}"'] & \mathrm{Fun}(\mathcal{C}, \mathcal{C}). \end{tikzcd}$$

Thus there is a unique section $s_{\Sigma} : \mathcal{C} \to \mathcal{C}^{\Sigma}$. So now we can define $\Sigma : \mathcal{C} \to \mathcal{C}$ to be

$$\Sigma : \mathcal{C} \xrightarrow{s_{\Sigma}} \mathcal{C}^{\Sigma} \xrightarrow{\sim} \mathcal{C}.$$

Analogously we can define Ω.

Theorem 4.3.7. If $\mathcal{C}$ is a pointed and finitely bicomplete category, we have an adjunction

$$\Sigma : \mathcal{C} \leftrightarrows \mathcal{C} : \Omega.$$

In particular, for $X, Y \in \mathcal{C}$ we have

$$\mathrm{Hom}_{\mathcal{C}}(\Sigma X, Y) \simeq \Omega \,\mathrm{Hom}_{\mathcal{C}}(X, Y).$$

This is because maps from $\Sigma X \to Y$ are in bijection with

$$\begin{tikzcd} \Omega \,\mathrm{Hom}(X, Y) \arrow[r] \arrow[d] & \mathrm{Hom}(0, Y) \arrow[d] \\ \mathrm{Hom}(0, Y) \arrow[r] & \mathrm{Hom}(\Sigma X, Y). \end{tikzcd}$$

This tells us that

$$\pi_0 \operatorname{Hom}_{\mathcal{C}}(\Sigma X, Y) = \pi_1 \operatorname{Hom}_{\mathcal{C}}(X, Y),$$

which is a group. Similarly we get that $\pi_0 \operatorname{Hom}(\Sigma^2 X, Y)$ is an abelian group.

Definition 4.3.8. Given $f : X \to Y$ in $\mathcal{C}$, we can define the *fiber* and *cofiber* as

$$\begin{tikzcd} \operatorname{fib}(f) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d] \\ * \arrow[r] & Y \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ * \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \operatorname{cof}(f) \end{tikzcd}$$

Definition 4.3.9. An ∞-category is *stable* if it is

- pointed
- finitely bicomplete
- triangles are exact if and only if they are coexact.

This last condition is equivalent to any of the following

- a square is a pullback iff it is a pushout
- $\Sigma : \mathcal{C} \leftrightarrows \mathcal{C} : \Omega$ is an equivalence
- $\operatorname{cof} : \operatorname{Fun}(\Delta^1, \mathcal{C}) \to \operatorname{Fun}(\Delta^1, \mathcal{C}) : \operatorname{cof}$ is an equivalence.

Let $\mathcal{C}$ be a stable ∞-category. Then

$$\pi_0 \operatorname{Hom}(X, Y) \cong \pi_0(\operatorname{Hom}(\Sigma X', Y)) \cong \pi_0 \operatorname{Hom}(\Sigma^2 X'', Y)$$

for some X, X''. Thus $\operatorname{Ho}(\mathcal{C})$ is an additive category.

We furthermore have that $\operatorname{Ho}(\mathcal{C})$ is triangulated. Given $f : X \to Y$ in $\mathcal{C}$,

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & 0 \arrow[d] \\ Y \arrow[r] \arrow[d] \arrow[ur, phantom, "\ulcorner"] & \operatorname{cof}(f) \arrow[d] \\ 0 \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \Sigma X \end{tikzcd}$$

Example 4.3.10. $\mathcal{C} = D(R)$. Show this has all the properties mentioned above.

Given $\mathcal{C}$ pointed, we want it to be stable. We can force $\Omega : \mathcal{C} \to \mathcal{C}$ to be an equivalence by considering

$$\operatorname{Sp}(\mathcal{C}) := \lim \left(\cdots \xrightarrow{\Omega} \mathcal{C} \xrightarrow{\Omega} \mathcal{C} \right).$$

Historically, we tried to invert Σ (Freudenthal theorem).

We could take $\operatorname{Sp}^{\text{naive}}$, whose objects are finite pointed spaces, and morphisms are stable maps $[X, Y]$. The problem is that Σ is not an equivalence on this category.

We could instead take $\operatorname{Sp}^{\text{Wh}}$, where objects are pairs (X, n) with X a pointed finite CW complex, and

$$\operatorname{Hom}((X, n), (Y, m)) := \operatorname{colim}_k \left[\Sigma^{n+k} X, \Sigma^{m+k} Y \right].$$

Then we have

$$\begin{aligned} \mathrm{Sp}^{\mathrm{naive}} &\hookrightarrow \mathrm{Sp}^{\mathrm{Wh}} \\ X &\mapsto (X, 0). \end{aligned}$$

The suspension takes the form

$$\begin{aligned} \Sigma : \mathrm{Sp}^{\mathrm{Wh}} &\to \mathrm{Sp}^{\mathrm{Wh}} \\ (X, n) &\mapsto (X, n+1). \end{aligned}$$

Thus

$$\mathrm{Sp}^{\mathrm{Wh}} = \operatorname{colim}\left(\mathcal{S}_*^{\mathrm{fin}} \xrightarrow{\Sigma} \mathcal{S}_*^{\mathrm{fin}} \xrightarrow{\Sigma} \cdots\right).$$

and we have that

$$\begin{aligned} \mathrm{Sp}(\mathcal{S}_*) &= \mathrm{Ind}(\mathrm{Sp}^{\mathrm{Wh}}) \\ &= \mathrm{Ind}\operatorname{colim}\left(\mathcal{S}_*^{\mathrm{fin}} \xrightarrow{\Sigma} \mathcal{S}_*^{\mathrm{fin}} \xrightarrow{\Sigma} \cdots\right) \\ &= \lim\left(\mathrm{Ind}(\mathcal{S}_*^{\mathrm{fin}}) \xleftarrow{\Omega} \mathrm{Ind}(\mathcal{S}_*^{\mathrm{fin}}) \xleftarrow{\Omega} \cdots\right) \\ &= \lim\left(\mathcal{S}_* \xleftarrow{\Omega} \mathcal{S}_* \xleftarrow{\Omega} \cdots\right). \end{aligned}$$

Note that $\operatorname{colim}\left(\mathcal{S}_* \xrightarrow{\Sigma} \mathcal{S}_* \xrightarrow{\Sigma} \cdots\right)$ won't work.

Definition 4.3.11. If $\mathcal{C}$ is a pointed finitely bicomplete ∞-category, a *prespectrum* in $\mathcal{C}$ is defined to be a functor

$$N(\mathbb{Z} \times \mathbb{Z}) \to \mathcal{C},$$

where $X_{i,j} = 0$ for $i \neq j$. Note that we get induced structure maps $\alpha_n : \Sigma X_n \to X_{n+1}$ and $\beta_n : X_n \to \Omega X_{n+1}$.

A prespectrum is called a *spectrum* in $\mathcal{C}$ if β_n's are equivalences for all n. We define $\mathrm{Sp}(\mathcal{C})$ to be the full subcategory of spectra.

Let

$$\mathrm{Sp}(\mathcal{C}) \simeq \lim\left(\mathcal{C} \xleftarrow{\Omega} \mathcal{C} \xleftarrow{\Omega} \cdots\right).$$

If $\mathcal{C} = \mathcal{S}_*$, we will write $\mathrm{Sp} = \mathrm{Sp}(\mathcal{S}_*)$ as the ∞-category of spectra. We define $\mathrm{Ho}(\mathrm{Sp})$ to be the stable homotopy category.

There is a functor

$$\widetilde{\Sigma}^\infty : \mathcal{C} \to \mathrm{PSp}(\mathcal{C}),$$

given by sending X to the prespectrum whose (i,i)th entry is $\Sigma^i X$.

Then there is a functor for $\mathcal{C}$ presentable

$$\mathrm{PSp}(\mathcal{C}) \to \mathrm{Sp}(\mathcal{C})$$

sending a prespectrum X to $\widetilde{X}$, defined by

$$\widetilde{X}_n := \operatorname{colim}(X_n \xrightarrow{\beta_n} \Omega X_{n+1} \to \cdots).$$

Then $\widetilde{X}_n \simeq \operatorname{colim}_k \Omega^k X_{n+k} \simeq \operatorname{colim}_k \Omega^{k+1} X_{n+k+1}$. As Ω is a right adjoint it commutes with filtered colimits (using presentable here), so this can be rewritten as

$$\Omega \operatorname{colim}_k \Omega^k X_{n+k+1} \simeq \Omega \widetilde{X}_{n+1}.$$

4.4 Multiplicative structure in spectra

Last time we had a universal property for $\mathcal{C} \xrightarrow{\Sigma^\infty} \mathrm{Sp}(\mathcal{C})$, where $\mathcal{C}$ was a pointed presentable ∞-category. We had that

$$\mathrm{Fun}^L(\mathrm{Sp}(\mathcal{C}), \mathcal{D}) \xrightarrow{\sim} \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

for any stable presentable ∞-category $\mathcal{D}$.

We denote by $\Sigma^\infty S^0 =: \mathbb{S} \in \mathrm{Sp} = \mathrm{Sp}(\mathcal{S}_*)$, and recall that

$$\mathrm{Fun}^L(\mathrm{Sp}, \mathcal{D}) \simeq \mathrm{Fun}^L(\mathcal{S}_*, \mathcal{D}) \simeq \mathcal{D}.$$

So we call Sp the free stable ∞-category generated by ∞.

Q: Can we give a symmetric monoidal structure on Sp analogous to $\otimes_{\mathbb{Z}}$ in Ab?

Spanier-Whitehead category: Recall Freudenthal says that if X and Y are finite CW complexes, then the sequence $[\Sigma^k X, \Sigma^k Y]$ stabilizes in k. So $\mathrm{Sp}^{\mathrm{naive}}$ has objects given by finite CW complexes, and homs given by stable maps.

To invert Σ, we introduced $\mathrm{Sp}^{\mathrm{Wh}}$, where objects are (X, n) and homs $(X, n) \to (Y, m)$ are

$$\mathrm{colim}_k \left[\Sigma^{n+k} X, \Sigma^{m+k} Y\right].$$

Formally in ∞-categories, we have that

$$\mathrm{Sp}^{\mathrm{Wh}} = \mathrm{colim}\left(\mathrm{Sp}_*^{\mathrm{fin}} \xrightarrow{\Sigma} \mathrm{Sp}_*^{\mathrm{fin}} \xrightarrow{\Sigma} \cdots\right).$$

Then $\mathrm{Sp} \simeq \mathrm{Ind}(\mathrm{Sp}^{\mathrm{Wh}})$.

Why finiteness? By adjunction we can see

$$\begin{aligned}\mathrm{Hom}((X,0),(Y,0)) &= \mathrm{colim}_k \left[\Sigma^k X, \Sigma^k Y\right] \\ &= \mathrm{colim}_k \left[X, \Omega^k \Sigma^k Y\right] \\ &= [X, \mathrm{colim}_k \Omega^k \Sigma^k Y],\end{aligned}$$

which holds if X is compact (e.g. finite CW). Thus if $\{-,-\}$ is a hom for spectra, we would have

$$\{X, Y\} = \{X, \Omega^n \Sigma^n Y\}.$$

What is the monoidal structure on $\mathrm{Sp}^{\mathrm{Wh}}$? Recall in $\mathcal{S}_*$ we have a smash product, so we could define

$$(X, n) \wedge (Y, m) := (X \wedge Y, n + m).$$

The unit is $(S^0, 0)$. This smash product is difficult to translate to spectra however.

Definition 4.4.1. For all $X \in \mathrm{Sp}$, we define

$$\pi_n(X) = \mathrm{Hom}_{\mathrm{Ho(Sp)}}(\Sigma^n \mathbb{S}, X) =: [\Sigma^n \mathbb{S}, X] \in \mathsf{Ab}.$$

In particular if X is a suspension spectrum, we get

$$\begin{aligned}\pi_k(\Sigma^\infty X) &= [\Sigma^n \mathbb{S}, \Sigma^\infty X] \\ &= \mathrm{colim}_k \left[\Sigma^{n+k} S^0, \Sigma^k X\right] \\ &= \mathrm{colim}_k \pi_{n+k}(X) \\ &= \pi_n^s(X).\end{aligned}$$

This is the *stable homotopy group* of X. It gives us a functor

$$\begin{aligned} \mathrm{Sp} &\to N(\mathsf{Ab}) \\ X &\mapsto \pi_n(X). \end{aligned}$$

This factors through

$$\mathrm{Sp} \xrightarrow{\Omega^\infty} \mathsf{S}_* \xrightarrow{\pi_n} N(\mathsf{Ab})$$

for $n \geq 2$.

(HA 1.4.3.8) The collection of these functors reflect equivalences. That is, if $\pi_n(X) \xrightarrow{\sim} \pi_n(Y)$ for all n, then $X \xrightarrow{\sim} Y$ in Sp.

Definition 4.4.2. We define $\mathrm{Sp}^{\geq 0}$ to be the ∞-category of *connective spectra*, the full subcategory of Sp on those X for which $\pi_n(X) = 0$ for $n < 0$.

Example 4.4.3. For all $X \in \mathcal{S}_*$, we have that $\Sigma^\infty X \in \mathrm{Sp}^{\geq 0}$.

We get an adjunction

$$\mathrm{Sp}^{\geq 0} \leftrightarrows \mathrm{Sp} : \tau_{\geq 0},$$

where the right adjoint to the inclusion is the *connective cover*.

If $X \in \mathcal{S}_*$ and $Y \in \mathrm{Sp}^{\geq 0}$, we have that

$$[\Sigma^\infty X, Y] \simeq [X, Y_0].$$

That is, $\Omega^\infty Y \simeq Y_0$.

If $Y \in \mathrm{Sp}^{\geq 0}$ then $\Omega^\infty Y = Y_0$ is an infinite loop space. That is, for all $k \geq 0$, we have that $Y_0 \simeq \Omega^k Y_k$. May recognition tells us that

$$\mathrm{Alg}^{\mathrm{gplike}}_{E_\infty}(\mathcal{S}_*) \simeq \mathrm{Sp}^{\geq 0}.$$

If $\mathcal{C}$ is a symmetric monoidal category, then $\mathrm{CAlg}(\mathcal{C})$ is also a sym mon cat with *some* underlying tensor product.

For example if X, Y are E_∞-algebras which are grouplike in spaces, then $X \wedge Y$ is an E_∞-algebra in $\mathcal{S}_*$. It is *not true* that if X and Y are infinite loop spaces then $X \wedge Y$ is an infinite loop space.

Example 4.4.4. Let G be an abelian group, then $K(G, 0)$ is an ∞-loop space, with $K(G, 0) \simeq \Omega^n K(G, n)$. Let $HG \in \mathrm{Sp}^{\geq 0}$ be its corresponding spectrum, called the *Eilenberg-Maclane spectrum* of G. This gives a functor

$$\begin{aligned} N(\mathsf{Ab}) &\to \mathrm{Sp}^{\geq 0} \\ G &\mapsto HG. \end{aligned}$$

We want a monoidal structure on Sp and $\mathrm{Sp}^{\geq 0}$ for this functor to be compatible with $\otimes_{\mathbb{Z}}$ in Ab.

Ideas for monoidal structure on Sp:

- On $\mathrm{Sp}^{\mathrm{Wh}}$ we had $(X, n) \wedge (Y, m) = (X \wedge Y, n + m)$
- $\mathrm{Alg}_{E_\infty}(\mathcal{S}_*)$
- $\mathsf{Ab}, \otimes_{\mathbb{Z}}$

Boardman: We could define $(X \wedge Y)_n = X_{a(n)} \wedge Y_{b(n)}$ where $a(n) + b(n) = n$, and then we could "Ω-spectrify." There are lots of choices for $a(n)$ and $b(n)$.

Adams: We could define

$$(X \wedge Y)_n \simeq \bigvee_{e_{ij}} \Sigma^{n-i-j-d} X_i \wedge Y_j \wedge M(\tau) / \sim$$

where e_{ij} is the square on the $\mathbb{Z} \times \mathbb{Z}$ grid with bottom left corner based at (i, j), open on the top and right sides, and $M(\tau)$ is the Thom complex of a bundle over e_{ij}.

Indexing on $\mathbb{Z} \times \mathbb{Z}$ is hard because we need to understand choices. Model categories allow us to switch $\mathbb{Z} \times \mathbb{Z}$ to something that records the choices.

Symmetric spectra: we get a model category Sp^Σ indexed on finite sets and injective morphisms (Hovey-Shipley-Smith).

Orthogonal spectra: (or EKMM spectra) $\mathrm{Sp}^{\mathcal{O}}$, indexed on real inner product spaces. This is by Mandell-May-Schwede-Shipley.

Theorem 4.4.5. (Lewis, '91) There is no good 1-category Sp^1 that describes Sp with a monoidal structure so that:

1. Sp^1 is symmetric monoidal
2. There is an adjunction $\Sigma^\infty : \mathrm{Top}_* \leftrightarrows \mathrm{Sp}^1 : \Omega^\infty$
3. We have that $\Sigma^\infty S^0$ is the unit
4. Ω^∞ is lax symmetric monoidal
5. For any pointed space, $\Omega^\infty \Sigma^\infty X \simeq \mathrm{colim}_k\, \Omega^k \Sigma^k X$. (that is, these functors are really doing stabilization of spaces)

For symmetric and orthogonal spectra, it is (3) that messes up — you really need a fibrant replacement. In EKMM they force (3) to be true, but fail (5).

How to think of $X \wedge Y$ in Sp? We use the universal properties, and try to understand its homotopy groups. There is a Künneth spectra sequence to compute $\pi_n(X \wedge Y)$.

Recall that $\mathrm{Fun}^L(\mathcal{S}, \mathcal{C}) \simeq \mathcal{C}$ for $\mathcal{C}$ any presentable ∞-category. This should remind us of the statement that $\mathrm{Hom}_R(R, M) = M$ for M an R-module. So we want to think of $\mathrm{Fun}^L(-,-)$ as an internal hom somewhere.

Definition 4.4.6. Let Pr^L denote the (very large) ∞-category of presentable ∞-categories, where

$$\mathrm{Hom}_{\mathrm{Pr}^L}(\mathcal{C}, \mathcal{D}) := \mathrm{Fun}^L(\mathcal{C}, \mathcal{D}).$$

Fact 4.4.7. This is an internal hom — i.e. if $\mathcal{C}$ and $\mathcal{D}$ are presentable, then $\mathrm{Fun}^L(\mathcal{C}, \mathcal{D})$ is presentable.[3]

We have that

$$\mathrm{Fun}^L(\mathcal{C}_1, \mathrm{Fun}^L(\mathcal{C}_2, \mathcal{D})) \simeq \mathrm{Fun}^{BL}(\mathcal{C}_1 \times \mathcal{C}_2, \mathcal{D}),$$

that is, functors $\mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{D}$ which are colimit preserving in each variable.

So we want some tensor product so that the above is equivalent to $\mathrm{Fun}^L(\mathcal{C}_1 \otimes \mathcal{C}_2, \mathcal{D})$.

Fact 4.4.8. If $\mathcal{C}$ is closed monoidal, then $\mathcal{C}^{\mathsf{op}}$ becomes closed monoidal, but where the tensor product and hom switch roles.

[3] If we took Fun instead of Fun^L, the size might increase, but in fact $\mathrm{Fun}^L(\mathcal{C}, \mathcal{D})$ is presentable in the same size sense that m$\mathcal{C}$ and $\mathcal{D}$ are.

The op of Pr^L is Pr^R, where we take limit-preserving functors! So we can check that

$$\mathcal{C}_1 \otimes \mathcal{C}_2 \simeq \mathrm{Fun}^R(\mathcal{C}_1^{\mathsf{op}}, \mathcal{C}_2).$$

By construction $\mathcal{S}$ is the monoidal unit, since

$$\begin{aligned} \mathcal{S} \otimes \mathcal{C} &= \mathrm{Fun}^R(\mathcal{S}^{\mathsf{op}}, \mathcal{C}) \\ &= \left(\mathrm{Fun}^L(\mathcal{S}, \mathcal{C}^{\mathsf{op}})\right)^{\mathsf{op}} \\ &= (\mathcal{C}^{\mathsf{op}})^{\mathsf{op}} \\ &= \mathcal{C}. \end{aligned}$$

Here we are using that

$$\mathrm{Fun}^R(-,-) = \mathrm{Fun}^L(-^{\mathsf{op}}, -^{\mathsf{op}})^{\mathsf{op}}.$$

So we need to create our operator category $\left(\mathrm{Pr}^L\right)^{\otimes} \subseteq \mathsf{Cat}_\infty^{\otimes} \simeq N(\mathrm{sSet}^{\otimes})[W_{\mathrm{Joyal}}^{-1}]$. We had a cocartesian fibration $\mathsf{Cat}_\infty^{\otimes} \to \mathrm{Fin}_*$, and we're going to restrict fibers to get the correct thing. The fibers will look like $(\mathcal{C}_1, \ldots, \mathcal{C}_n)$ with $\mathcal{C}_i$ presentable, and appropriate morphisms.

So the construction we just did argues that $\mathrm{Pr}^L \hookrightarrow \mathsf{Cat}_\infty$ is a lax symmetric monoidal functor. Then

$$\mathrm{Alg}_{E_\infty}(\overset{L}{\mathrm{Pr}}) = \{\text{presentably symmetric monoidal } \infty\text{-cats}\},$$

and $\mathcal{S}$ is the initial object. This provides the universal property of spaces with its monoidal structure $\mathcal{S} \times \mathcal{S} \to \mathcal{S}$, colimit-preserving in each variable, with the point as the unit.

4.5 Brown Representability

We've seen that the monoidal product on spectra has two intuitions:

1. $\mathrm{Sp}^{\geq 0} \simeq \mathrm{Alg}_{E_\infty}^{\mathrm{gplike}}(\mathcal{S}_*)$
2. $\mathrm{Sp}^{\mathrm{Wh}} = \mathrm{colim}\left(\mathcal{S}_*^{\mathrm{fin}} \xrightarrow{\Sigma} \cdots\right)$.[4] This had a smash product.

Recall $(\mathcal{S}, \times, *)$ was the initial object in $\mathrm{Alg}_{E_\infty}\left(\mathrm{Pr}^L\right)$. We saw we had

$$\left(\overset{L}{\mathrm{Pr}}, \otimes \mathcal{S}\right) \to \left(\mathsf{Cat}_\infty, \times, \Delta^0\right),$$

with tensor $\mathcal{C} \otimes \mathcal{D} = \mathrm{Fun}^R(\mathcal{C}^{\mathsf{op}}, \mathcal{D})$ and internal hom $\mathrm{Fun}^L(\mathcal{C}, \mathcal{D})$.

We have $\mathsf{Cat}_\infty^{\mathrm{st}} \subseteq \mathsf{Cat}_\infty$ on stable ∞-categories and exact functors, and a corresponding $\mathrm{Pr}_{\mathrm{st}}^L \subseteq \mathrm{Pr}^L$ spanned by stable ∞-categories.

The stabilization functor $\mathcal{C} \mapsto \mathrm{Sp}(\mathcal{C})$ can be viewed as left adjoint to the inclusion

$$\mathrm{Sp} : \overset{L}{\mathrm{Pr}} \leftrightarrows \underset{\mathrm{st}}{\overset{L}{\mathrm{Pr}}}$$

Tensoring with spectra, we get

$$\begin{aligned} \mathcal{C} \otimes \mathrm{Sp} &= \mathrm{Fun}^R(\mathcal{C}^{\mathsf{op}}, \mathrm{Sp}) \\ &= \mathrm{Fun}^R(\mathcal{C}^{\mathsf{op}}, \lim(\mathcal{S}_* \leftarrow \cdots)) \\ &= \lim\left(\mathrm{Fun}^R(\mathcal{C}^{\mathsf{op}}, \mathcal{S}_*) \leftarrow \cdots\right) \\ &= \lim(\mathcal{C}_* \leftarrow \cdots) \\ &= \mathrm{Sp}(\mathcal{C}). \end{aligned}$$

[4] We have that $\mathcal{S}_*^{\mathrm{fin}}$ is finite CW complexes, *not* the compact objects in $\mathcal{S}_*$.

Fact: If $\mathcal{C}, \mathcal{D}$ stable then $\mathrm{Fun}^L(\mathcal{C}, \mathcal{D}) \in \mathrm{Pr}^L_{\mathrm{st}}$.

We can think of stabilization as "extension of scalars" along $\mathcal{S}_* \xrightarrow{\Sigma^\infty} \mathrm{Sp}$. We have a monoidal adjunction

$$\left(\overset{L}{\mathrm{Pr}}, \otimes, \mathcal{S}\right) \leftrightarrows \left(\underset{\mathrm{st}}{\overset{L}{\mathrm{Pr}}}, \otimes, \mathrm{Sp}\right).$$

Recall Sp is the initial object in $\mathrm{Pr}^L_{\mathrm{st}}$. This characterizes spectra together with

$$\mathrm{Sp} \times \mathrm{Sp} \xrightarrow{\wedge} \mathrm{Sp}$$

monoidal and bicolimit preserving so that $\mathbb{S}$ is the unit.

We have that $\Sigma^\infty_+ : \mathcal{S} \to \mathrm{Sp}$ is strong monoidal, and $\Omega^\infty : \mathrm{Sp} \to \mathcal{S}$ is lax monoidal, implying that

$$\Sigma^\infty_+ X \wedge \Sigma^\infty_+ Y \simeq \Sigma^\infty_+ (X \times Y).$$

We can also shift

$$\Sigma^{\infty - k} : \mathcal{S}_* \leftrightarrows \mathrm{Sp} : \Omega^{\infty - k}.$$

We call $E_k = \Omega^{\infty - k} E$.

Formula: For any $E \in \mathrm{Sp}$, we have that

$$\begin{aligned} E &\simeq \mathrm{colim}_k\, \Sigma^{\infty-k}\Omega^{\infty-k} E \\ &\simeq \mathrm{colim}_k\, \Sigma^{\infty-k} E_k. \end{aligned}$$

For $E, F \in \mathrm{Sp}$

$$\begin{aligned} E \wedge F &= (\mathrm{colim}_a\, \Sigma^{\infty-a} E_a) \wedge \left(\mathrm{colim}_b\, \Sigma^{\infty-b} F_b\right) \\ &= \mathrm{colim}_{a,b}\, \Sigma^{\infty-a-b} E_a \wedge F_b. \end{aligned}$$

Example 4.5.1. Recall Mayer-Vietoris: for $U, V \subseteq X$ open, we have an LES

$$\cdots \to H_*(U \cap V) \to H_*(U) \oplus H_*(V) \to H_*(U \cup V) \to H_{*-1}(U \cap V) \to \cdots$$

Recall that $H_*(X) = H_*(C_*(X))$, and by Dold-Kan, we have that $C_*(X) = \pi_* \mathbb{Z}[\mathrm{Sing}(X)]$. Let's reinterpret Mayer-Vietoris in this setting. It is saying that there is a homotopy pullback in sSet of the form

$$\begin{tikzcd} \mathbb{Z}\mathrm{Sing}_*(U \cap V) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{Z}\mathrm{Sing}_* U \arrow[d] \\ \mathbb{Z}\mathrm{Sing}_* V \arrow[r] & \mathbb{Z}\mathrm{Sing}_* U \cup V. \end{tikzcd}$$

We can view homology as

$$\begin{aligned} \mathrm{CW}^{\mathrm{fin}}_* &\to \mathrm{Kan} \\ X &\mapsto \mathbb{Z}\mathrm{Sing}_* X. \end{aligned}$$

Mayer-Vietoris is the statement that this sends homotopy pushouts to homotopy pullbacks. We can view this functor as $\mathcal{S}^{\mathrm{fin}}_* \to \mathcal{S}$.

Q: Can we do this for all homology theories?

Definition 4.5.2. (Eilenberg-Steenrod) A (reduced) homology theory is $\left\{\widetilde{E}_n : \mathrm{CW}^{\mathrm{fin}}_* \to \mathsf{Ab}\right\}$ such that

1. $\widetilde{E}_n$ invariant under homotopy
2. Excision: $\widetilde{E}^{i+1}(\Sigma X) \cong \widetilde{E}_i(X)$
3. Additivity: $\widetilde{E}_i(X \vee Y) \cong \widetilde{E}_i(X) \oplus \widetilde{E}_i(Y)$
4. Exactness: if $f : X \to Y$ then

$$\widetilde{E}_n(X) \to \widetilde{E}_n(Y) \to \widetilde{E}_n(Cf).$$

Goal: We can view $\widetilde{E}_* : \mathrm{CW}_* \to \mathsf{Ab}$ as a certain $\widetilde{E} : \mathcal{S}_*^{\mathrm{fin}} \to \mathcal{S}$.
Axiom (1) allows us to extend $\widetilde{E}_*$ to $\mathrm{Ho}(\mathcal{S}_*^{\mathrm{fin}})$. Axiom (2) comes from

$$C_*(\Sigma X)[-1] \simeq_{\mathrm{qiso}} C_*(X).$$

If and only if $\Omega\mathbb{Z}\mathrm{Sing}\Sigma X \simeq \mathbb{Z}\mathrm{Sing}X$. So we're rephrasing that

$$\Omega\widetilde{E}(\Sigma X) \simeq \widetilde{E}(X).$$

Axiom (3) comes from $C_*(X \vee Y) \simeq C_*(X) \oplus C_*(Y)$. Translating this over to simplicial sets via Dold-Kan, we get

$$\mathbb{Z}\mathrm{Sing}X \vee Y \simeq \mathbb{Z}\mathrm{Sing}X \times \mathbb{Z}\mathrm{Sing}Y.$$

This gives $\widetilde{E}(X \vee Y) \cong \widetilde{E}(X) \oplus \widetilde{E}(Y)$ and hence $\pi_*(X \times Y) \cong \pi_*(X) \oplus \pi_*(Y)$.
(4) Says $\pi_i(\mathrm{fib}(f)) = \ker(\pi_i(f))$. We have that $C_*(X) \simeq \ker\left(C_*(Y) \to C_*(f)\right)$. Then

$$\mathbb{Z}\mathrm{Sing}_*(X) \xrightarrow{\sim} \mathrm{fib}\left(\mathbb{Z}\mathrm{Sing}Y \to \mathbb{Z}\mathrm{Sing}Cf\right).$$

Hence

$$\widetilde{E}(X) \simeq \mathrm{fib}\left(\widetilde{E}(Y) \to \widetilde{E}(Cf)\right).$$

That is,

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ * \arrow[r] & Cf \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

is sent to

$$\begin{tikzcd} \widetilde{E}(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \widetilde{E}(Y) \arrow[d] \\ * \arrow[r] & \widetilde{E}(Cf). \end{tikzcd}$$

Definition 4.5.3. Let $\mathcal{C}$ be an ∞-category. We say a functor $F : \mathcal{S}_*^{\mathrm{fin}} \to \mathcal{C}$ is

1. *excisive* if F sends pushouts to pullbacks
2. *reduced/pointed* if $F(*) = *$.

We write $\mathrm{Exc}_*(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{C}) \subseteq \mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{C})$ for excisive and reduced functors.

Given any $\mathcal{S}_*^{\mathrm{fin}} \xrightarrow{\widetilde{E}} \mathcal{S}$ excisive, we obtain a reduced homology theory

$$\mathsf{S}_*^{\mathrm{fin}} \xrightarrow{\widetilde{E}} \mathsf{S} \xrightarrow{\pi_*^s} \mathsf{Ab}.$$

Theorem 4.5.4. There is an equivalence

$$\mathrm{Sp}(\mathcal{C}) \simeq \mathrm{Exc}_*(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{C}).$$

Proof. For some $\widetilde{E} \in \mathrm{Exc}_*(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{S})$, we want to define $E \in \mathrm{Sp}$. We define $E_0 = \widetilde{E}(S^0)$, and $E_1 = \widetilde{E}(S^1)$, etc. We can define $E_{-n} = \Omega^n E_0$. This works because

$$\begin{tikzcd} S^n \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & S^{n+1} \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

is sent to

$$\begin{tikzcd} \widetilde{E}(S^n) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & * \arrow[d] \\ * \arrow[r] & \widetilde{E}(S^{n+1}). \end{tikzcd}$$

This gives maps $\widetilde{E}(S^n) \xrightarrow{\sim} \Omega\widetilde{E}(S^{n+1})$.

For the other direction, given $E \in \mathrm{Sp}$, we can get an excisive functor sending

$$X \mapsto X \wedge E_0.$$

We can reinterpet

$$\begin{aligned} \Omega^\infty : \mathrm{Exc}_*(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{S}) &\to \mathcal{S} \\ \widetilde{E} &\mapsto \widetilde{E}(S^0). \end{aligned}$$

We can show this is universal. ◻

Given E a spectrum, we have an associated reduced homology theory where $\widetilde{E}_*(X) := \pi_*^s X \wedge E_0$.

We have that

$$\Sigma_+^\infty X \wedge E = \mathrm{colim}_k\, \Sigma^{\infty - k} X \wedge E_k.$$

We have that

$$\begin{aligned} \Pi_*(\Sigma^\infty X \wedge E) &= \pi_* \left(\mathrm{colim}_k\, \Sigma^{\infty-k} X \wedge E_k\right) \\ &= \mathrm{colim}_k\, \pi_* \left(\Sigma^{\infty-k} X \wedge E_k\right) \\ &= \mathrm{colim}_k\, \pi_{*+k}(X \wedge E_k). \end{aligned}$$

This is exactly the definition of $\pi_*^s(X \wedge E_0)$.

Thus

$$\widetilde{E}_*(X) = \pi_* \Sigma^\infty X \wedge E.$$

Example 4.5.5. Sphere spectrum $\mathbb{S} \in \mathrm{Sp}$ gives the functor

$$\begin{aligned} \mathcal{S}_*^{\mathrm{fin}} &\to \mathcal{S} \\ X &\mapsto X \wedge QS^0, \end{aligned}$$

where $Q(-) = \Omega^\infty \Sigma^\infty (-) = \mathrm{colim}_k\, \Omega^k \Sigma^k (-)$. Model categorically they think of this as just the natural inclusion $\mathcal{S}_*^{\mathrm{fin}} \to \mathcal{S}$ because they derive after including. The homology theory is $\widetilde{\mathbb{S}}_* = \pi_*^s(-)$.

Example 4.5.6. We have that $\widetilde{H\mathbb{Z}}_*(X) = H_*(X;\mathbb{Z})$. Dold-Thom lets us relate $\Sigma^\infty X \wedge H\mathbb{Z}$ with $\mathbb{Z}\mathrm{Sing}(X)$ somehow.

Definition 4.5.7. For F a spectrum, we can define

$$\widetilde{E}_*(F) = \pi_*(E \wedge F).$$

Theorem 4.5.8. (Brown representability) If $\widetilde{E}_*(-) : \mathrm{CW}_*^{\mathrm{fin}} \to \mathsf{Ab}$ is a reduced homology theory, then there exists $E \in \mathrm{Sp}$ such that $\widetilde{E}_n(X) = \pi_n^S(X \wedge E_0)$.

We looked at π_* of $E \wedge -$. Taking the same thing for its adjoint, we call $F(E,-)$ the right adjoint to $E \wedge -$ (this exists because colimit-preserving + presentable). Can take the internal hom to be

$$F(E,E')_n = \mathrm{Hom}_{\mathrm{Sp}}(E, \Sigma^n E').$$

Can define $\widetilde{E}^n(X) = [X, E_n] = [\Sigma^{\infty-n}X, E]$. In fact, $F(E,E')_n = \mathrm{Hom}_{\mathrm{Sp}}(E,\Sigma^n E')$. This is because

$$\mathrm{Hom}_{\mathrm{Sp}}(E \wedge \mathbb{S}, F) \simeq \mathrm{Hom}_{\mathrm{Sp}}(\mathbb{S}, F(E,F)).$$

Think $\mathrm{Hom}_R(R,M) = M$ and $\mathrm{Hom}_{\mathrm{Ch}_R}(R, M_*) = M_0$. Then $\mathrm{Hom}_{\mathrm{Sp}}(\mathbb{S}, E) \simeq E_0$. This follows from the loops suspension adjunction:

$$\mathrm{Hom}(\Sigma^\infty_+ *, E) = \mathrm{Hom}(\mathbb{S}, E) = \mathrm{Hom}(*, \Omega^\infty E) = E_0.$$

For $E \in \mathrm{Sp}$ can define reduced associated cohomology theory for $X \in \mathcal{S}_*$

$$\widetilde{E}^n(X) = [X, E_n] = \pi_n F(\Sigma^\infty X, E).$$

4.6 Modules in spectra

Monoidal categories which are not symmetric:

- Let $\mathcal{C}$ be any category, and look at $\mathrm{End}(\mathcal{C})$ with composition and the identity
- G any non-abelian monoid, defines a discrete monoidal category.
- Bimodules over any non-commutative ring

Recall a sm ∞-cat was $\mathcal{C}^\otimes \to N(\mathrm{Fin}_*)$ a cocartesian fibration + Segal condition. This gave $N(\mathrm{Fin}_*) \to \mathsf{Cat}_\infty$.

We had $\tau : \langle 2 \rangle \to \langle 2 \rangle$, sending $0 \mapsto 0$ and swapping 1,2. This alone gave a symmetric structure on $\otimes$. We want to restrict from Fin_* to throw out τ and its friends.

There are multiple ways to do this: can view $\Delta^{\mathsf{op}} \subseteq \mathrm{Fin}_*$ sending $[n] \mapsto \langle n \rangle$. Given $\alpha : [k] \to [n]$ we send it to a map

$$\begin{aligned} \langle n \rangle &\to \langle k \rangle \\ j &\mapsto \begin{cases} i & \exists i\colon j \in [\alpha(i-1)+1, \alpha(i)] \\ * & \text{else} \end{cases} \end{aligned}$$

The composite

$$\Delta^{\mathsf{op}} \to \mathrm{Fin}_* \subseteq \mathsf{Set}_*$$

defines the pointed simplicial set $S^1 = \Delta^1/\partial\Delta^1 \in \mathrm{sSet}_*$.

Definition 4.6.1. A monoidal ∞-cat is a cocart fibration $\mathcal{C}^{\otimes} \to N(\Delta^{\mathsf{op}})$ with the Segal condition $\mathcal{C}^{\otimes}_{[n]} \to \left(\mathcal{C}^{\otimes}_{[1]}\right)^{\times n}$ given by cocartesian lifts of $p^i : [1] \to [n]$, $0 \mapsto i-1$, $1 \mapsto i$.

By straightening we get $N(\Delta^{\mathsf{op}}) \to \mathsf{Cat}_\infty$ sending $[n] \to \mathcal{C}^{\times n}$. This is some kind of bar construction.

Definition 4.6.2. $\alpha \in \Delta$ is inert if $\alpha : [n] \to [k]$ is injective, and $\mathrm{im}(\alpha) \subseteq [k]$ is convex. Inert things in Δ^{op} map to inert things in Fin_* under the map defined above.

Definition 4.6.3. A *lax monoidal functor* $F^{\otimes} : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ is a functor

$$\begin{tikzcd} \mathcal{C}^{\otimes} \arrow[rr, "F^{\otimes}"] \arrow[dr] & & \mathcal{D}^{\otimes} \arrow[dl] \\ & N(\Delta^{\mathsf{op}}), & \end{tikzcd}$$

so that $F^{\otimes}$ sends cocart lifts of inert to cocart lifts.

A lax monoidal functor $F^{\otimes}$ is one that sends *all* cocartesian lifts to cocartesian lifts.
Given $\mathcal{C}$ a monoidal ∞-cat, we have that

$$\mathrm{Alg}_{E_1}(\mathsf{C}) = \mathrm{Fun}^{\mathrm{lax}}_{E_1}(N(\Delta^{\mathsf{op}}), \mathcal{C}^{\otimes}).$$

Every symmetric monoidal ∞-cat can be viewed as a monoidal ∞-cat via

$$\begin{tikzcd} \widetilde{\mathcal{C}}^{\otimes} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{C}^{\otimes} \arrow[d] \\ N(\Delta^{\mathsf{op}}) \arrow[r] & N(\mathrm{Fin}_*). \end{tikzcd}$$

We could also straighten then precompose with $N(\Delta^{\mathsf{op}}) \to N(\mathrm{Fin}_*)$.
To define modules over a ring, we will use the bar construction $[n] \mapsto N \otimes R^{\otimes n} \otimes M$.

Definition 4.6.4. Let $p : \mathcal{C}^{\otimes} \to N(\Delta^{\mathsf{op}})$ be a monoidal ∞-cat. An ∞-cat $\mathcal{M}$ is said to be *left tensored over* $\mathcal{C}$ if there is a cocart $q : \mathcal{E} \to N(\Delta^{\mathsf{op}})$ so that

$$\begin{tikzcd} \mathcal{E} \arrow[rr, "f"] \arrow[dr, "q"'] & & \mathcal{C}^{\otimes} \arrow[dl, "p"] \\ & N(\Delta^{\mathsf{op}}). & \end{tikzcd}$$

that sends cocart lifts to cocart lifts, such that

$$\mathcal{E}_{[n]} \xrightarrow{\sim} \mathcal{C}^{\otimes}_{[n]} \times \mathcal{E}^{\otimes}_{\{n\}}$$

for $\{n\} \subseteq [n]$, with $\mathcal{M} = \mathcal{E}_{[0]}$, and $\mathcal{E}_{[1]} \simeq \mathcal{C} \times \mathcal{M}$. This is formalizing a functor $\mathcal{C} \times \mathcal{M} \to \mathcal{M}$ compatible with monoidal structure on $\mathcal{C}$.

Example 4.6.5. $\mathcal{C}$ is left tensored over itself. Then $\mathcal{E} = \widetilde{\mathcal{C}}^{\otimes}$ with $\mathcal{E}_{[n]} = \mathcal{C}^{\times(n+1)}$.

Definition 4.6.6. Given $\mathcal{M}$ left tensored over $\mathcal{C}$, a left module of $\mathcal{M}$ is a map $s : N(\Delta^{\mathsf{op}}) \to \mathcal{M}^{\otimes}$ such that

$$N(\Delta) \xrightarrow{s} \mathcal{M}^{\otimes} \xrightarrow{f} \mathcal{C}^{\otimes}$$

is a lax monoidal functor (if $\alpha : [k] \to [n]$ is inert in Δ then $f(\alpha)$ is a cocart fib of $\mathcal{C}^{\otimes}$).

We write $\mathrm{LMod}(\mathcal{M}) \subseteq \mathrm{Fun}_{N(\Delta^{\mathsf{op}})}(N(\Delta^{\mathsf{op}}), \mathcal{M}^{\otimes})$ spanned by left modules. We are interested in when $\mathcal{M} = \mathcal{C}$. In that case

$$\begin{aligned} \mathrm{LMod}(\mathcal{C}) &\to \mathrm{Alg}_{E_1}(\mathcal{C}) \\ (\mathcal{M}, \mathcal{A}) &\mapsto \mathcal{A}. \end{aligned}$$

Given $A \in \mathrm{Alg}_{E_1}(\mathcal{C})$, we can define A-modules as

$$\begin{tikzcd} {}_A\mathrm{Mod}(\mathcal{C}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{LMod}(\mathcal{C}) \arrow[d] \\ * \arrow[r, "A"'] & \mathrm{Alg}_{E_1}(\mathcal{C}). \end{tikzcd}$$

Can define left A-modules and (A, A)-bimodules in a similar way.

Can defgine left modules in a sym mon ∞-cat

$$\begin{tikzcd} \mathrm{LMod}_{E_\infty}(\mathcal{C}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{LMod}(\mathcal{C}) = \mathrm{LMod}_{E_1}(\mathcal{C}) \arrow[d] \\ \mathrm{Alg}_{E_\infty}(\mathcal{C}) \arrow[r] & \mathrm{Alg}_{E_1}(\mathcal{C}). \end{tikzcd}$$

Can check that if $A \in \mathrm{Alg}_{E_\infty}(\mathcal{C})$ then ${}_A\mathrm{Mod}(\mathcal{C}) \cong \mathrm{Mod}_A(\mathcal{C})$.

4.7 The Schwede–Shipley Theorem

Goal: generalize the Freyd-Mitchell and Gabriel theeorems.

Given a lax monoidal functor $F : \mathcal{C} \to \mathcal{D}$ between (symmetric) monoidal ∞-cats, it induces a functor

$$\begin{aligned} \mathrm{Alg}_{E_1}(\mathcal{C}) &\to \mathrm{Alg}_{E_1}(\mathcal{D}) \\ A &\mapsto F(A). \end{aligned}$$

1-categorically if $A \in \mathcal{C}$ is an associative algebra, then $F(A)$ is an associative algebra.

So lax monoidal is the correct notion needed to lift algebras.

∞-categorically, $\mathrm{Alg}_{E_1}(\mathcal{C})$ are lax monoidal functors $* \to \mathcal{C}$. The statement follows from the fact that composition of lax monoidal functors is lax monoidal.

If F is lax *symmetric* monoidal, then we can lift

$$F : \mathrm{Alg}_{E_\infty}(\mathcal{C}) \to \mathrm{Alg}_{E_\infty}(\mathcal{D}).$$

We also have, for all $A \in \mathrm{Alg}_{E_1}(\mathcal{C})$,

$$F : \mathrm{Mod}_A(\mathcal{C}) \to \mathrm{Mod}_{F(A)}(\mathcal{D}).$$

Recall

$$\begin{aligned} N(\mathsf{Ab}) &\to \mathrm{Sp} \\ A &\mapsto HA. \end{aligned}$$

Here

1. $(HA)_n = K(A, n)$ for $n > 0$ and $*$ for $n < 0$
2. $HA : \mathcal{S}_*^{\mathrm{fin}} \to \mathcal{S}$ sends $X \mapsto X \wedge K(A, 0)$

3. By Brown representability, $\widetilde{H}^n(X, A) \cong [X, K(A, n)]$.

4. $\pi_n(HA) = A$ if $n = 0$ and 0 otherwise

$HA \in \mathrm{Sp}^{\geq 0}$ then the associated element in $\mathrm{Alg}^{\mathrm{gplike}}_{E_\infty}(\mathcal{S}_*)$ is A as a discrete pointed space.
Given $A, B \in \mathsf{Ab}$ we can compare $HA \wedge HB$ with $A \otimes_{\mathbb{Z}} B$. These are not the same.

$$\pi_0 HA \wedge HB \cong A \otimes_{\mathbb{Z}} B$$
$$\pi_n HA \wedge HB \neq 0 \qquad n > 0.$$

If $A = B = \mathbb{F}_2$, then

$$\pi_* \left(H\mathbb{F}_2 \wedge H\mathbb{F}_2\right) = \mathbb{F}_2[\xi_1, \xi_2, \ldots]$$

with $|\xi_i| = 2^i - 1$. This is the *dual Steenrod algebra.*

We can get a map $HA \wedge HB \to H(A \otimes_{\mathbb{Z}} B)$ by adjunction

$$\pi_0 : \mathrm{Sp}^{\geq 0} \leftrightarrows N(\mathsf{Ab}) : H(-).$$

Then

$$\pi_0(E \wedge F) \cong \pi_0(E) \otimes_{\mathbb{Z}} \pi_0(F).$$

Thus π_0 is strong symmetric monoidal.

Exercise 4.7.1. If $L : \mathcal{C} \leftrightarrows \mathcal{D} : R$ is an adjunction between symmetric monoidal categories, if L is strong monoidal then R is lax monoidal.

Warning: $\pi_0 : \mathrm{Sp} \to N(\mathsf{Ab})$ is not strong monoidal on the entire category of spectra.

Since the inclusion $\mathrm{Sp}^{\geq 0} \hookrightarrow \mathrm{Sp}$ is lax symmetric monoidal, we have the composite $N(\mathsf{Ab}) \xrightarrow{H} \mathrm{Sp}^{\geq 0} \to \mathrm{Sp}$ is, hence we get

$$N(\mathsf{Ring}) = \mathrm{Alg}_{E_1}(N(\mathsf{Ab})) \to \mathrm{Alg}_{E_1}(\mathrm{Sp})$$
$$R \mapsto HR.$$

We call $\mathrm{Alg}_{E_1}(\mathrm{Sp})$ *ring spectra.*

Can we compare with $\mathsf{Ab} = \mathrm{Mod}_{\mathbb{Z}} \to D(\mathbb{Z})$? Yes we can view $D(\mathbb{Z})$ as $\mathrm{Mod}_{H\mathbb{Z}}(\mathrm{Sp})$ in a monoidal way.

Recall that for $R \in \mathrm{CRing}$, we get $D(R) = N(\mathrm{Ch}_R)[W^{-1}_{\mathrm{proj}}]$, which is symmetric monoidal ∞-cat with $\otimes^{\mathbb{L}}_R$. We want a monoidal structure on $\mathrm{Mod}_{HR}(\mathrm{Sp})$ that mimics the derived tensor product.

Recall 1-categorically that $R \in \mathrm{Alg}(\mathcal{C}, \otimes, I)$ and $M \in \mathrm{Mod}_R(\mathcal{C})$ and $N \in {}_R\mathrm{Mod}(\mathcal{C})$, we define $\otimes_R$ by the coequalizer

$$M \otimes R \otimes N \rightrightarrows M \otimes N \to M \otimes_R N.$$

So on spectra we want a *relative smash product.*

We have to kill off much higher terms

$$M \wedge_{HR} N := \mathrm{colim}\left(\cdots \rightrightarrows M \wedge HR^{\wedge 2} \wedge N \rightrightarrows M \wedge HR \wedge N \rightrightarrows M \wedge N\right)$$

More generally, given $R \in \mathrm{Alg}_{E_1}(\mathcal{C})$, we can define $M \otimes_R N$ as the colimit of a bar construction. In a 1-category the higher maps don't matter and we just recover the coequalizer definition.

We have

$$N(\Delta^{\mathsf{op}}) \to N(\mathrm{Fin}_*) \to \mathcal{C} \hookrightarrow \mathsf{Cat}_\infty,$$
$$[n] \mapsto M \otimes R^{\otimes n} \otimes N.$$

For $\mathcal{C} = \mathrm{Sp}$, this defines $(\mathrm{Mod}_R(\mathrm{Sp}), \wedge_R, R)$. We can also define $F_R(M, -) : \mathrm{Mod}_R(\mathrm{Sp}) \to \mathrm{Mod}_R(\mathrm{Sp})$ to be the right adjoint of

$$M \wedge_R - : \mathrm{Mod}_R \to \mathrm{Mod}_R.$$

Notation 4.7.2. If $R \in \mathrm{Alg}_{E_\infty}(\mathrm{Sp})$, and $M, N \in \mathrm{Mod}_R(\mathrm{Sp})$, we can define

$$\mathrm{Tor}_*^R(M, N) := \pi_* (M \wedge_R N)$$
$$\mathrm{Ext}_R^*(M, N) := \pi_{-*} F_R(M, N).$$

We shall see that

$$\pi_* (HM \wedge_{HR} HN) \cong \mathrm{Tor}_*^R(M, N),$$

where $R \in \mathrm{CAlg}(\mathsf{Ab})$ and $M, N \in \mathrm{Mod}_R(\mathsf{Ab})$.

We have *change of algebras*: if $f : A \to B$ in $\mathrm{Alg}_{E_\infty}(\mathcal{C})$, we get a monoidal adjunction

$$- \otimes_A B : \mathrm{Mod}_A \leftrightarrows \mathrm{Mod}_B : f^*,$$

where extension is strong monoidal and restriction f^* is lax.

In spectra this becomes

$$- \wedge R : \mathrm{Mod}_{\mathbb{S}} = \mathrm{Sp} \leftrightarrows \mathrm{Mod}_R : U.$$

Theorem 4.7.3. *(Schwede-Shipley)* Let $\mathcal{C}$ be a stable ∞-category. Then $\mathcal{C} \simeq \mathrm{Mod}_R \mathrm{Sp}$ if and only if $\mathcal{C}$ is presentable, and there exists $C \in \mathcal{C}$ compact generator such that if $D \in \mathcal{C}$ and $\mathrm{Ext}^n_{\mathcal{C}}(C, D) \cong 0$ then $D \simeq 0$.

Lemma 4.7.4. If $\mathcal{C}$ is a stable ∞-category, and $X, Y \in \mathcal{C}$, then $\mathrm{Hom}_{\mathcal{C}}(X, Y) \in \mathrm{Sp}$.

Proof. We have that $\mathrm{Hom}_{\mathcal{C}}(X, Y) \in \mathcal{S}_*$, so

$$\Omega \, \mathrm{Hom}_{\mathcal{C}}(X, Y) \simeq \mathrm{Hom}_{\mathcal{C}}(\Sigma X, Y) \simeq \mathrm{Hom}_{\mathcal{C}}(X, Y).$$

So these are infinite loop spaces. □

Proof of theorem: if $\mathcal{C} \simeq \mathrm{Mod}_R(\mathrm{Sp})$, then $\mathcal{C}$ is presentable. Take $C = R$, then $\mathrm{Ext}^n_{\mathcal{C}}(R, D) \cong \pi_n D$. Then $D \simeq 0$ if and only if $\pi_{-n} D = 0$ for all n.

For the other direction, if $\mathcal{C} \in \mathrm{Pr}^L$, then as $\mathcal{C}$ is stable, there is a map

$$\mathrm{Sp} \otimes \mathcal{C} \to \mathcal{C}$$
$$(E, C) \mapsto E \otimes C,$$

adjoint to $\mathrm{Hom}_{\mathcal{C}}(C, -)$ valued in Sp. That is, $\mathcal{C}$ is tensored and cotensored over spectra.

We have

$$- \otimes C : \mathrm{Sp} \leftrightarrows \mathcal{C} : \mathrm{Hom}_{\mathcal{C}}(C, -).$$

Let $G = \mathrm{Hom}_{\mathcal{C}}(C, -)$, then the idea is that this is monadic and the monad is equivalent to $- \wedge_{\mathbb{S}} R$ for some R.

Let $\alpha : D \to D'$ in $\mathcal{C}$ such that $G(\alpha)$ is an equivalence in Sp. Then $G(C\alpha) \simeq 0$.

$$\pi_n C\alpha \simeq \mathrm{Ext}^{-n}_{\mathcal{C}}(C, G\alpha) = 0,$$

so $C\alpha \simeq 0$, so α equivalence in $\mathcal{C}$.

Then $R := G(C) = \mathrm{Hom}_{\mathcal{C}}(C, C) = \mathrm{End}_{\mathcal{C}}(C) \in \mathrm{Alg}_{E_1}(\mathrm{Sp})$.

With $E \in \mathrm{Sp}$ and $D \in \mathcal{C}$, get $E \wedge G(D) \simeq G(E \otimes D)$. This is true as G preserves all colimits, suffices to check for $E = \mathbb{S}$ then obvious. $R = G(C)$, $E \wedge R = G(X \otimes C)$, Barr Beck Lurie monadicity.

If $R = \mathrm{End}_{\mathcal{C}}(C)$ get an monoidal variant

$$\mathrm{Alg}_{E_\infty}(\mathrm{Sp}) \to \mathrm{Alg}_{E_\infty}(\overset{L}{\mathrm{Pr}})$$
$$R \mapsto \mathrm{Mod}_R.$$

So we can say that $\mathcal{C} \in \mathrm{Alg}_{E_\infty}(\mathrm{Pr}^L)$ belongs to the image above if and only if there is some $I \in \mathcal{C}$ a compact generator.

Theorem 4.7.5. *(Stable Dold Kan)* Let R be a commutative ring. Then

$$(\mathrm{Mod}_{HR}(\mathrm{Sp}), \wedge_R, HR) \simeq \left(D(R), \otimes_R^{\mathbb{L}}, R\right).$$

Proof sketch. Take $D_* \in \mathrm{Ch}_R$, then $H_n(D_*) = \mathrm{Ext}_R^{-n}(R, D_*) \cong \mathrm{Ext}_{D(R)}^{-n}(R, D_*)$. Thus R is a compact generator.

Thus $D(R) \simeq \mathrm{Mod}_A(\mathrm{Sp})$, where $A = \mathrm{End}_{D(R)}(R)$, but we check

$$\begin{aligned}\pi_n(A) &\cong \mathrm{Ext}_{D(R)}^{-n}(R, R) \\ &= \begin{cases} R & n = 0 \\ 0 & \text{else,} \end{cases}\end{aligned}$$

so $A \simeq HR$. □

Shipley proved this in model categories in early 2000's.

4.8 Universal trace methods for algebraic K-theory

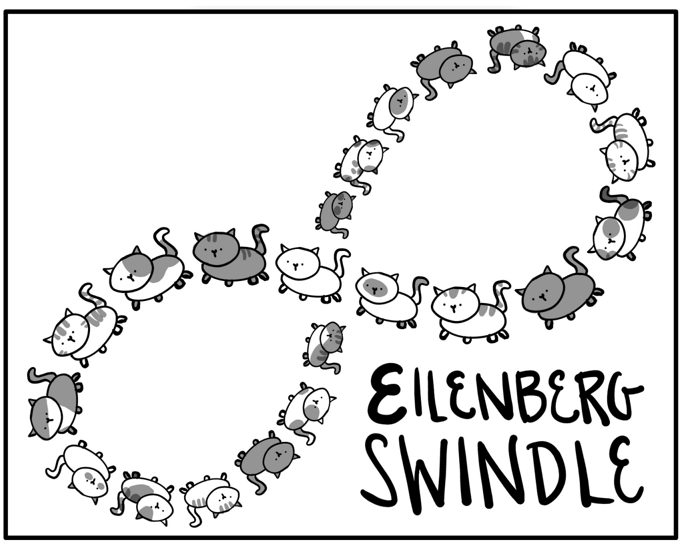
Recall: for $R \in \mathsf{Ring}$, we can define $K_0(R) = K_0(\mathcal{P}(R))$. The latter K_0 is Grothendieck group completion of commutative monoids, and here $\mathcal{P}(R)$ is iso classes of finitely generated projective (right) R-modules.

If $M \oplus N \cong R^n$, then $[M] + [N] = [R^n]$ in $K_0(R)$. That is, exact sequences split in $K_0(R)$.

Eilenberg swindle: If we just did projective, not also finitely generated, we would get 0 because any projective M has $M \oplus N \cong R^n$ for some N, n, hence we could take

$$R^\infty = M \oplus N \oplus M \oplus N \oplus \cdots$$

Since $M \oplus R^\infty \cong R^\infty$, this would imply $[M] = 0$.



Definition 4.8.1. $K_n(R) = \pi_n(\mathrm{BGL}(R)^+ \times K_0(R))$.

Here $\mathrm{GL}(R) = \mathrm{colim}_n \, \mathrm{GL}_n(R)$, and the plus construction is the universal H-space receiving a map from $\mathrm{BGL}(R)$, abelianizing π_1, ...

Note that $\mathrm{BGL}(R)^+ \times K_0(R)$ is an infinite loop space. It admits a Gersten-Wagoner delooping.

K-theory can be generalized to a much wider context, e.g. exact categories, and stable ∞-categories.

For example R corresponds to the stable ∞-category $\mathrm{Mod}_{HR}^{\mathrm{cpct}}(\mathrm{Sp})$. Taking compact objects is again to avoid size issues.

Blumberg-Gepner-Tabuada: Define connective K-theory as a functor

$$\mathsf{Cat}_\infty^{\mathrm{st}} \to \mathrm{Sp}^{\geq 0},$$

where $\mathsf{Cat}_\infty^{\mathrm{st}}$ is the category of stable ∞-categories and exact functors (preserves finite limits and colimits).

Definition 4.8.2. Let $\mathsf{Cat}_\infty^{\mathrm{perf}} \subseteq \mathsf{Cat}_\infty^{\mathrm{st}}$ be the full subcategory spanned by idempotent-complete categories.

We have that $\mathcal{C}$ is idempotent complete if for all $X \in \mathcal{C}$, and any $e : X \to X$ in $\mathcal{C}$ such that $e^2 \simeq e$, we have a splitting onto its image.

F.g. projective modules are idempotent complete, free modules are not.

Idempotent completion is a left adjoint to the inclusion:

$$\mathrm{Idem} : \mathsf{Cat}_\infty^{\mathrm{st}} \leftrightarrows \mathsf{Cat}_\infty^{\mathrm{perf}} : i.$$

We have that $\mathrm{Idem}(\mathcal{C}) = \mathrm{Ind}(\mathcal{C})^\omega$ (BGT 2.20).

Think of an idempotent complete stable ∞-category as the compact objects of a presentable stable ∞-category.

To define $K(\mathcal{C})$ for $\mathcal{C} \in \mathsf{Cat}_\infty^{\mathrm{perf}}$, we can take

$$K(\mathcal{C}) = |S_\bullet \mathcal{C}^\simeq|.$$

K-theory is comprised of two concepts:

- abelian group completion
- splitting exact sequences

Definition 4.8.3. Let $\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}$ be exact functors btw stable ∞-categories.

1. Say F is a *Morita equivalence* if $\mathrm{Idem}(F)$ is an equivalence of ∞-categories
2. The sequence is *exact* if F is fully faithful, $G \circ F \simeq 0$, and $\mathcal{C} \simeq \mathcal{B}/\mathcal{A}$ in $\mathsf{Cat}_\infty^{\mathrm{perf}}$.
3. The sequence is *split exact* if there exist right adjoint functors F', G' to F, G, respectively, so that $F'F = \mathrm{id}$ and $GG' = \mathrm{id}$.

Definition 4.8.4. Let $E : \mathsf{Cat}_\infty^{\mathrm{st}} \to \mathcal{D}$ where $\mathcal{D} \in \mathrm{Pr}_{\mathrm{st}}^L$. We say E is *additive* if it:

1. inverts Morita equivalences
2. preserves filtered colimits
3. sends split exact sequences to split (co)fiber sequences in $\mathcal{D}$, i.e. $E(\mathcal{B}) \simeq E(\mathcal{A}) \vee E(\mathcal{B})$.

Take

$$\mathsf{Cat}_\infty^{\mathrm{st}} \xrightarrow{\mathrm{Idem}} \mathsf{Cat}_\infty^{\mathrm{perf}} \hookrightarrow \mathrm{Fun}\left(\left(\mathsf{Cat}_\infty^{\mathrm{perf}}\right)^{\mathsf{op}}, \mathcal{S}\right) \xrightarrow{\mathrm{Sp}} \mathrm{Fun}(\left(\mathsf{Cat}_\infty^{\mathrm{perf}}\right)^{\mathsf{op}}, \mathrm{Sp}) \to \mathrm{Fun}(\left(\mathsf{Cat}_\infty^{\mathrm{perf}}\right)^{\mathsf{op}}, \mathrm{Sp})/\sim$$

where we mod out by split exact sequences.

We call the resulting object $\mathcal{M}_{\mathrm{add}}$, and the composition

$$\mathcal{U}_{\mathrm{add}} : \mathsf{Cat}_\infty^{\mathrm{st}} \to \mathcal{M}_{\mathrm{add}}.$$

This functor is the universal additive invariant, in the sends that

$$\mathrm{Fun}^L(\mathcal{M}_{\mathrm{add}}, \mathcal{D}) \xrightarrow{\mathcal{U}_{\mathrm{add}}^*} \mathrm{Fun}_{\mathrm{add}}(\mathsf{Cat}_\infty^{\mathrm{st}}, \mathcal{D})$$

for all $\mathcal{D} \in \mathrm{Pr}_{\mathrm{st}}^L$.

Definition 4.8.5. For $\mathcal{C} \in \mathsf{Cat}_\infty^{\mathrm{perf}}$, we define

$$K(\mathcal{C}) = \mathrm{Hom}_{\mathcal{M}_{\mathrm{add}}}(\mathcal{U}_{\mathrm{add}}(\mathrm{Sp}^{\mathrm{Wh}}), \mathcal{U}_{\mathrm{add}}(\mathcal{C})) \in \mathrm{Sp}^{\geq 0}.$$

We can make this universal property monoidal: if $\mathcal{C}$ is a symmetric monoidal ∞-category, then $K(\mathcal{C})$ is an E_∞ ring spectrum.

Can construct $\otimes$ in $\mathsf{Cat}_\infty^{\mathrm{perf}}$ similar to Pr^L. This induces a monoidal structure on $\mathrm{Fun}_{\mathrm{add}}(\mathsf{Cat}_\infty^{\mathrm{perf}}, \mathrm{Sp})$ by Day convolution.

Here $\mathcal{U}_{\mathrm{add}}$ is strong monoidal. Then for all $\mathcal{D} \in \mathrm{Alg}_{E_\infty}(\mathrm{Pr}^L)$, we get

$$\mathrm{Fun}^{L,lax}(\mathcal{M}_{\mathrm{add}}, \mathcal{D}) \simeq \mathrm{Fun}^{lax}_{\mathrm{add}}(\mathsf{Cat}_\infty^{\mathrm{perf}}, \mathcal{D}).$$

Application: Dennis trace $K(R) \to \mathrm{THH}(R)$. Here $\mathrm{THH}(R) = R \wedge_{R \wedge R^{\mathrm{op}}} R$. If R is a k-algebra, get

$$\mathbb{H}_*(R) = H_*(R \otimes^{\mathbb{L}}_{R \otimes R^{\mathrm{op}}} R) = \mathrm{Tor}_*^{R \otimes R^{\mathrm{op}}}(R, R).$$

Can replace R by any stable ∞-cat $\mathcal{C}$. Here

$$\mathrm{THH}(\mathcal{C}) = \mathrm{colim}\left(\cdots \amalg_{(c_0,\ldots,c_n)} \mathcal{C}(c_{n-1}, c_n) \wedge \cdots \wedge \mathcal{C}(c_n, c_0)\right).$$

Here $\mathrm{THH}(\mathrm{Mod}_R^{\mathrm{perf}}) = \mathrm{THH}(R)$.

We have $\mathrm{THH} : \mathsf{Cat}_\infty^{\mathrm{st}} \to \mathrm{Sp}^{\geq 0}$. It is an additive invariant (clearly preserves Morita equivalence and filtered colimits). Can use Dennis-Waldhausen-Morita argument to show it sends split exact sequences to cofiber sequences.

Theorem 4.8.6. Let E be any additive invariant, i.e. $E \in \mathrm{Fun}_{\mathrm{add}}(\mathsf{Cat}_\infty^{\mathrm{st}}, \mathrm{Sp})$. Then $\mathrm{Nat}\,(K, E) \simeq E(\mathrm{Sp}^{\mathrm{Wh}})$.

We see that

$$\mathrm{Nat}\,(K(-), \mathrm{THH}(-)) \cong \mathrm{THH}(\mathrm{Sp}^{\mathrm{Wh}}) \simeq \mathrm{THH}(\mathbb{S}) \simeq \mathbb{S}.$$

Applying π_0, we get that

$$[K(\mathcal{C}), \mathrm{THH}(\mathcal{C})] \cong \pi_0\mathbb{S} \cong \mathbb{Z}.$$

Given $F : K(\mathcal{C}) \to \mathrm{THH}(\mathcal{C})$, we get

$$\mathbb{S} \to \mathrm{Map}(\mathcal{U}_{\mathrm{add}}(\mathrm{Sp}^{\mathrm{Wh}}), \mathcal{U}_{\mathrm{add}}(\mathrm{Sp}^{\mathrm{Wh}})) \simeq K(\mathbb{S}) \xrightarrow{F} \mathrm{THH}(\mathbb{S}) \simeq \mathbb{S}.$$

The Dennis trace picks up $1 \in \mathbb{Z}$.

We can view $K(R) \to \mathrm{THH}(R)$ via

$$\mathrm{BGL}_n(R) \to B^{\mathrm{cyc}}\mathrm{GL}_n(R) \to B^{\mathrm{cyc}}M_n(R) \to B^{\mathrm{cyc}}R.$$

On π_0, we get

$$K_0(R) \to HH_0(R).$$

For $R \in \mathsf{Ring}$, we send $[P] \mapsto \mathrm{tr}(\mathrm{id}_P \oplus 0)$.

Bibliography